



Further improvements for SAT in terms of formula length

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ABSTRACT

In this paper, we prove that the general CNF satisfiability problem can be solved in $O^*(1.0638^L)$ time, where L is the length of the input CNF-formula (i.e., the total number of literals in the formula), which improves the previous result of $O^*(1.0652^L)$ obtained in 2009. Our algorithm was analyzed by using the measure-and-conquer method. Our improvements are mainly attributed to the following two points: we carefully design branching rules to deal with degree-5 and degree-4 variables to avoid previous bottlenecks; we show that some worst cases will not always happen, and then we can use an amortized technique to get further improvements. In our analyses, we provide some general frameworks for analysis and several lower bounds on the decreasing of the measure to simplify the arguments. These techniques may be used to analyze more algorithms based on the measure-and-conquer method.

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1. Introduction

Propositional Satisfiability is the problem of determining, for a formula of the propositional calculus, if there is an assignment of truth values to its variables for which that formula evaluates to true. By SAT, we mean the problem of propositional satisfiability for formulas in conjunctive normal form (CNF) [1]. SAT is the first problem proved to be NP-complete [2], and it plays an important role in computational complexity, artificial intelligence, and many others [3]. There are numerous investigations on this problem in different fields, such as approximation algorithms, randomized algorithms, heuristic algorithms, and exact and parameterized algorithms. In this paper, we study parameterized algorithms for SAT parameterized by the input length.

In order to measure the running time bound for SAT, there are three frequently used parameters: the number of variables n , the number of clauses m , and the input length L . The input length L is defined as the sum of the number of literals in each clause. The number of variables n should be the most basic parameter. The simple brute force algorithm to try all 2^n possible assignments of the n variables will get the running time bound of $O^*(2^n)$.¹ After decades of hard work, no one can break this trivial bound. The Strong Exponential Time Hypothesis (SETH) conjectures that SAT cannot be solved in time $O^*(c^n)$ for some constant $c < 2$ [4]. For a restricted version, the k -SAT problem (the length of each clause in the formula is bounded by a constant k) can be solved in $O^*(c(k)^n)$ time for some value $c(k) < 2$ depending on k . There are two major research lines: one is deterministic local search algorithms like Schöning's algorithm [5,6] and the other one is randomized algorithms based on PPZ [7] or PPSZ [8,9]. For example, 3-SAT can be deterministically solved in $O^*(1.32793^n)$ time [6] and 4-SAT can be deterministically solved in $O^*(1.49857^n)$ time [6].

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¹ The O^* notation supervises all polynomial factors, i.e., $f(n) = O^*(g(n))$ means $f(n) = O(g(n)n^{O(1)})$.

Table 1
Previous and our upper bounds for SAT.

Running time bounds	References
$O^*(1.0927^L)$	Van Gelder 1988 [14]
$O^*(1.0801^L)$	Kullmann and Luckhardt 1997 [15]
$O^*(1.0758^L)$	Hirsch 1998 [11]
$O^*(1.0740^L)$	Hirsch 2000 [16]
$O^*(1.0663^L)$	Wahlström 2005 [17]
$O^*(1.0652^L)$	Chen and Liu 2009 [18]
$O^*(1.0638^L)$	This paper

When it comes to the parameter m , Monien *et al.* [10] first gave an algorithm with time complexity $O^*(1.260^m)$ in 1981. Later, the bound was improved to $O^*(1.239^m)$ by Hirsch [11] in 1998, and then improved to $O^*(1.234^m)$ by Yamamoto [12] in 2005. Now the best result is $O^*(1.2226^m)$ obtained by Chu, Xiao, and Zhang [13].

The input length L is another important and frequently studied parameter. It is probably the most precise parameter to describe the size of the input CNF-Formula. From the first algorithm with running time bound $O^*(1.0927^L)$ by Van Gelder [14] in 1988, the result was improved several times. In 1997, the bound was improved to $O^*(1.0801^L)$ by Kullmann and Luckhardt [15]. Later, the bound was improved to $O^*(1.0758^L)$ by Hirsch [11] in 1998, and improved again by Hirsch [16] to $O^*(1.0740^L)$ in 2000. Then Wahlström [17] gave an $O^*(1.0663^L)$ -time algorithm in 2005. In 2009, Chen and Liu [18] achieved a bound $O^*(1.0652^L)$ by using the measure-and-conquer method. In this paper, we further improve the result to $O^*(1.0638^L)$. We list the major progress and our result in Table 1.

It is also worth mentioning the Maximum Satisfiability problem (MaxSAT), which is strongly related to the SAT problem. SAT asks whether we can give an assignment of the variables to satisfy all clauses, while MaxSAT asks us to satisfy the maximum number of clauses. For MaxSAT, researchers usually consider the decision version of it: whether we can satisfy at least k clauses. Thus, k is a natural parameter in parameterized algorithms. The current best result for MaxSAT parameterized by k is $O^*(1.325^k)$ [19]. The three parameters n , m , and L mentioned above are also frequently considered for MaxSAT. For the number of variables n , similar to SAT, the bound $O^*(2^n)$ obtained by a trivial algorithm is still not broken so far, and it is impossible to break under SETH. In terms of the number of clauses m , the running time bound of MaxSAT was recently improved to $O^*(1.2886^m)$ [20]. When it comes to the length of the formula L , very recently, the bound was also improved to $O^*(1.0911^L)$ [21].

Our algorithm, as well as most algorithms for SAT-related problems, is based on the branch-and-search paradigm. The fundamentals of branching heuristics about the SAT problem can be found in [3]. The idea of the branch-and-search algorithm is simple and practical: for a given CNF-formula $\mathcal{F}$, we iteratively branch on a variable or literal x into two branches by assigning value 1 or 0 to it. Let $\mathcal{F}_{x=1}$ and $\mathcal{F}_{\bar{x}=1}$ be the resulting CNF-formulas by assigning value 1 and 0 to x , respectively. It holds that $\mathcal{F}$ is satisfiable if and only if at least one of $\mathcal{F}_{x=1}$ and $\mathcal{F}_{\bar{x}=1}$ is satisfiable. To get a running time bound, we need to analyze how much the parameter L can decrease in each branch. To break bottlenecks in direct analysis, some references [17,18] analyzed the algorithms based on new measures and gave the relation between the new measures and L . The measure-and-conquer method is one of the frequently used techniques. Our algorithm in this paper will also adopt the measure-and-conquer method and deal with variables from high degree to low degree. We compare our algorithm with the previous algorithm [18] based on the measure-and-conquer method. The algorithm in [18] carefully analyzed branching operations for variables of degree 4 and used a simple and uniform rule to deal with variables of degree at least 5. Their bottlenecks are some cases to deal with degree-4 variables. Our algorithm will carefully analyze the branching operation for degree-5 variables and use simple rules for degree-4 variables. In the conference version of this paper [22], we have shown that these modifications led to improvements. In this version, we further show that the bottlenecks will not always happen, and we can use an amortized technique to combine bottleneck cases with the following good cases to get a further improved average bound. To do this, we also need to modify some steps of the algorithm in the conference version [22] and carefully design and analyze the branching operations for some special structures to satisfy the requirement of amortization. Finally, we are able to improve the result to $O^*(1.0638^L)$. In our algorithm, to simplify some case analyses and arguments, we provide some general frameworks for analysis and establish several lower bounds on the decreasing of the measure. The lower bounds may reveal some structural properties of the problem, and the analysis framework may be used to analyze more algorithms based on the measure-and-conquer method.

2. Preliminaries

Let $V = \{x_1, x_2, \dots, x_n\}$ denote a set of n boolean variables. Each variable x_i ($i \in \{1, 2, \dots, n\}$) has two corresponding literals: the positive literal x_i and the negative literal $\bar{x}_i$ (we use $\bar{x}$ to denote the negation of a literal x , and $\bar{\bar{x}} = x$). A clause on V consists of some literals on V . Note that we allow a clause to be empty. A clause containing literal $z_1, z_2, \dots, z_q$ is simply written as $z_1 z_2 \dots z_q$. Thus, we use zC to denote the clause containing literal z and all literals in clause C . We also use $C_1 C_2$ to denote the clause containing all literals in clauses C_1 and C_2 . We use $\bar{C}$ to denote a clause that contains the negation of every literal in clause C . That is, if $C = z_1 z_2 \dots z_q$, then $\bar{C} = \bar{z}_1 \bar{z}_2 \dots \bar{z}_q$. A CNF-formula on V is the conjunction of a set of

clauses $\mathcal{F} = C_1 \wedge C_2 \wedge \dots \wedge C_m$. When we say a variable x is contained in a clause (resp., a formula), it means that the clause (resp., at least one clause of the formula) contains the literal x or its negation $\bar{x}$.

An assignment for V is a map $A : V \rightarrow \{0, 1\}$. A clause C_j is *satisfied* by an assignment if and only if there exists at least one literal in C_j such that the assignment makes its value 1. A CNF-formula is *satisfied* by an assignment A if and only if each clause in it is satisfied by A . A CNF-formula is *satisfiable* if it can be satisfied by at least one assignment. We may assign value 0 or 1 to a literal, which is indeed to assign a value to its variable to make the corresponding literal 0 or 1.

A literal z is called an (i, j) -literal (resp., an (i^+, j) -literal or (i^-, j) -literal) in a formula $\mathcal{F}$ if z appears i (resp., at least i or at most i) times and $\bar{z}$ appears j times in the formula $\mathcal{F}$. Similarly, we can define (i, j^+) -literal, (i, j^-) -literal, (i^+, j^+) -literal, (i^-, j^-) -literal, and so on. Note that literal z is an (i, j) -literal if and only if literal $\bar{z}$ is a (j, i) -literal. A variable x is an (i, j) -variable if the positive literal x is an (i, j) -literal.

For a variable or a literal x in formula $\mathcal{F}$, the *degree* of it, denoted by $\deg(x)$, is the number of x appearing in $\mathcal{F}$ plus the number of $\bar{x}$ appearing in $\mathcal{F}$, i.e., $\deg(x) = i + j$ for an (i, j) -variable or (i, j) -literal x . A d -variable (resp., d^+ -variable or d^- -variable) is a variable with the degree exactly d (resp., at least d or at most d). The degree of a formula $\mathcal{F}$ is the maximum degree of all variables in $\mathcal{F}$. For a clause or a formula C , the set of variables whose literals appear in C is denoted by $\text{var}(C)$.

The *length* of a clause C , denoted by $|C|$, is the number of literals in C . A clause is a k -clause or k^+ -clause if the length of it is k or at least k . We use $L(\mathcal{F})$ to indicate the length of a formula $\mathcal{F}$. It is the sum of the lengths of all clauses in $\mathcal{F}$, which is also the sum of the degrees of all variables in $\mathcal{F}$. A formula $\mathcal{F}$ is called k -CNF formula if each clause in $\mathcal{F}$ has a length of at most k .

In a formula $\mathcal{F}$, a literal x is called a *neighbor* of a literal z if there is a clause containing both z and x . The set of neighbors of a literal z in a formula $\mathcal{F}$ is denoted by $N(z, \mathcal{F})$. We also use $N^{(k)}(x, \mathcal{F})$ (resp., $N^{(k^+)}(z, \mathcal{F})$) to denote the neighbors of z in k -clauses (resp., k^+ -clauses) in $\mathcal{F}$, i.e., for any $z' \in N^{(k)}(z, \mathcal{F})$ (resp., $z' \in N^{(k^+)}(z, \mathcal{F})$), there exists a k -clause (resp., k^+ -clause) containing both z and z' . If we say a variable y appears in $N(x, \mathcal{F})$, it means that literal y or literal $\bar{y}$ is in $N(x, \mathcal{F})$.

3. Some techniques

3.1. Branch-and-search algorithms

Our algorithm is a standard branch-and-search algorithm, which first applies some reduction rules to reduce the instance as much as possible and then searches for a solution by branching. The branching operations may exponentially increase the running time. We will use a measure to evaluate the size of the search tree generated in the algorithm. For the SAT problem, the number of variables or clauses of the formula is a commonly used measure.

Let μ be a measure of an instance. We use $T(\mu)$ to denote the number of leaves of the search tree generated by the algorithm for any instance with the measure being at most μ . For a branching operation that branches on the current instance into l branches with the measure decreasing by at least a_i in the i -th branch, we get a recurrence relation

$$T(\mu) \leq T(\mu - a_1) + T(\mu - a_2) + \dots + T(\mu - a_l).$$

The recurrence relation can also be simply represented by a *branching vector* $[a_1, a_2, \dots, a_l]$. The largest root of the function $f(x) = 1 - \sum_{i=1}^l x^{-a_i}$, denoted by $\tau(a_1, a_2, \dots, a_l)$, is called the *branching factor* of the recurrence. If the maximum branching factor for all branching operations in the algorithm is at most γ , then $T(\mu) = O(\gamma^\mu)$. If the algorithm runs in polynomial time on each node of the search tree, then the total running time of the algorithm is $O^*(\gamma^\mu)$. More details about analyzing recurrences can be found in the monograph [23].

In the analysis, we need to find the largest branching factor in the algorithm. Let $\mathbf{a} = [a_1, a_2, \dots, a_l]$ and $\mathbf{b} = [b_1, b_2, \dots, b_l]$ be two branching vectors. If $\tau(a_1, a_2, \dots, a_l) > \tau(b_1, b_2, \dots, b_l)$, then we say $\mathbf{a}$ *covers* $\mathbf{b}$. Usually, it is hard to compare two branching vectors directly. In this paper, we will frequently use some special cases. If it holds that $a_i \leq b_i$ for all $i = 1, 2, \dots, l$, then the branching factor of $\mathbf{b}$ is not smaller than that of $\mathbf{a}$, i.e., $\mathbf{a}$ covers $\mathbf{b}$. Let i and j be positive reals such that $0 < i < j$, for any $0 < \epsilon < \frac{j-i}{2}$, it holds that $\tau(i, j) > \tau(i + \epsilon, j - \epsilon)$. For this case, we have that the branching vector $[i, j]$ covers $[i + \epsilon, j - \epsilon]$.

3.2. Shift

In some cases, the worst branch in the algorithm will not always happen. In order to deal with this situation, one can use an amortization technique to get improved results. This technique has been used in several previous papers, see [24–27]. In this paper, we will follow the notation “shift” in [27] to implement the amortized analysis.

Consider two branching operations A and B with branching vectors $\mathbf{a} = [a_1, a_2]$ and $\mathbf{b} = [b_1, b_2]$ (with recurrences $T(\mu) \leq T(\mu - a_1) + T(\mu - a_2)$ and $T(\mu) \leq T(\mu - b_1) + T(\mu - b_2)$) such that the branching operation B has a smaller branching factor than A does, where $\mathbf{a}$ may be the bottleneck in the running time analysis of the algorithm ($\mathbf{a}$ has the maximum branching factor among all branching vectors). Suppose branching operation B is always applicable to the sub-instance $\mathcal{F}_1$ generated by the first sub-branch of A . In this case, we can get a better branching vector $\mathbf{c} = [a_1 + b_1, a_1 + b_2, a_2]$

by combining the branching operation A and the branching operation B applied to $\mathcal{F}_1$. Note that the branching operation B may also be applicable to the sub-instances generated by several branching operations other than A . In order to ease such an analysis without generating all combined branching vectors, we use a notion of “shift” [27]. We transfer some amount from the measure decreases in the recurrence for B to that for A as follows. We save an amount $\sigma > 0$ of measure decreases from B by evaluating the branching operation B with branching vectors

$$[b_1 - \sigma, b_2 - \sigma],$$

which leads to a larger branching factor than its original branching vector. The saved measure decrease σ will be included in the branching vectors for branching operation A to obtain

$$[a_1 + \sigma, a_2].$$

The saved amount σ is also called a *shift*, where the best value for σ will be determined so that the maximum branching factor is minimized. Clearly, we can get branching vector $[a_1 + b_1, a_1 + b_2, a_2]$ by combining the above two branching vectors.

3.3. Measure and conquer

The measure-and-conquer method [28] is a powerful tool for analyzing the branch-and-search algorithms. The main idea of the method is to adopt a new measure in the analysis of the algorithm. For example, instead of using the number of variables as the measure, it may set weights to different variables and use the sum of all variable weights as the measure. This method may be able to catch more structural properties and then get further improvements. Nowadays, the fastest exact algorithms for many NP-hard problems were designed by using this method [27,29,30]. In this paper, we will also use the measure-and-conquer method.

We introduce a weight to each variable in the formula according to the degree of the variable, $w: \mathbb{Z}^+ \rightarrow \mathbb{R}^+$, where $\mathbb{Z}^+$ and $\mathbb{R}^+$ denote the sets of nonnegative integers and nonnegative reals, respectively. Let w_i denote the weight of a variable with degree i . A variable with a lower degree will not receive a higher weight. i.e., $w_i \geq w_{i-1}$. In our algorithm, the measure of a formula $\mathcal{F}$ is defined as

$$\mu(\mathcal{F}) = \sum_x w_{\deg(x)}. \quad (1)$$

In other words, $\mu(\mathcal{F})$ is the sum of the weight of all variables in $\mathcal{F}$. Let n_i denote the number of i -variables in $\mathcal{F}$. Then we also have that

$$\mu(\mathcal{F}) = \sum_i w_i n_i.$$

One important step is to set the value of weight w_i . Different values of w_i will generate different branching vectors and factors. We need to find a good setting of w_i so that the worst branching factor is as small as possible. We will get the value of w_i by solving a quasiconvex program after listing all our branching vectors. However, we pre-specify some requirements of the weights to simplify arguments. Some similar assumptions were used in the previous measure-and-conquer analysis. We set the weight such that

$$\begin{aligned} w_1 &= w_2 = 0, \\ 0 &< w_3 < 2, w_4 = 2w_3, \text{ and} \\ w_i &= i \text{ for } i \geq 5. \end{aligned} \quad (2)$$

We use δ_i to denote the difference between w_i and w_{i-1} for $i > 0$, i.e., $\delta_i = w_i - w_{i-1}$. By (2), we have

$$w_3 = \delta_3 = \delta_4, \quad (3)$$

and $2w_5 > 5w_3 \Rightarrow 2w_5 - 4w_3 > w_3 \Rightarrow 2w_5 - 2w_4 > w_3$, which implies

$$2\delta_5 > w_3. \quad (4)$$

We also assume that

$$\begin{aligned} w_3 &\geq \delta_5, \\ 1 &\leq \delta_i \leq \delta_{i-1} \text{ for } i \geq 3, \text{ and} \\ w_3 - \delta_5 &< 1. \end{aligned} \quad (5)$$

Under these assumptions, it holds that $w_i \leq i$ for each i . Thus, for any formula $\mathcal{F}$, it always holds that

$$\mu(\mathcal{F}) \leq L(\mathcal{F}). \quad (6)$$

This tells us that if we can get a running time bound of $O^*(c^{\mu(\mathcal{F})})$ for a real number c , then we also get a running time bound of $O^*(c^{L(\mathcal{F})})$ for this problem. To obtain a running time bound in terms of the formula length $L(\mathcal{F})$, we consider the measure $\mu(\mathcal{F})$ and show how much the measure $\mu(\mathcal{F})$ decreases in the branching operations of our algorithm and find the worst branching factor among all branching vectors.

4. The algorithm

We will first introduce our algorithm and then analyze its running time bound by using the measure-and-conquer method. Our algorithm consists of reduction operations and branching operations. When no reduction operations can be applied anymore, the algorithm will search for a solution by branching. We first introduce our reduction rules.

4.1. Reduction rules

We have ten reduction rules. Most are well-known and frequently used in the literature (see [17,18] for examples). We introduce the reduction rules in the order as stated, and a reduction rule will be applied in our algorithm only when all the previous reduction rules do not apply to the instance.

R-Rule 1 (Elimination of duplicated literals). If a clause C contains duplicated literals z , remove all but one z in C .

R-Rule 2 (Elimination of subsumptions). If there are two clauses C and D such that $C \subseteq D$, remove clause D .

R-Rule 3 (Elimination of tautology). If a clause C contains two opposite literals z and $\bar{z}$, remove clause C .

R-Rule 4 (Elimination of 1-clauses and pure literals). If there is a 1-clause $\{x\}$ or a $(1^+, 0)$ -literal x , assign $x = 1$.

Davis-Putnam Resolution, proposed in [31], is a classic and frequently used technology for SAT. Let $\mathcal{F}$ be a CNF-formula and x be a variable in $\mathcal{F}$. Assume that clauses containing literal x are $xC_1, xC_2, \dots, xC_a$ and clauses containing literal $\bar{x}$ are $\bar{x}D_1, \bar{x}D_2, \dots, \bar{x}D_b$. A *Davis-Putnam resolution* on x is to construct a new CNF-formula $DP_x(\mathcal{F})$ by the following method: initially $DP_x(\mathcal{F}) = \mathcal{F}$; add new clauses $C_i D_j$ for each $1 \leq i \leq a$ and $1 \leq j \leq b$; and remove $xC_1, xC_2, \dots, xC_a, \bar{x}D_1, \bar{x}D_2, \dots, \bar{x}D_b$ from the formula. It is known that

Proposition 1 ([31]). A CNF-formula $\mathcal{F}$ is satisfiable if and only if $DP_x(\mathcal{F})$ is satisfiable.

In the resolution operation, each new clause $C_i D_j$ is called a *resolvent*. A resolvent is *trivial* if it contains both a literal and the negation of it. Since trivial resolvents will always be satisfied, we can simply delete trivial resolvents from the instance directly. So when we do resolutions, we assume that all trivial resolvents are deleted.

R-Rule 5 (Trivial resolution). If there is a variable x such that the degree of each variable in $DP_x(\mathcal{F})$ is not greater than that in $\mathcal{F}$, then apply resolution on x .

R-Rule 6 ([18]). If there are a 2-clause $z_1 z_2$ and a clause C containing both z_1 and $\bar{z}_2$, then remove $\bar{z}_2$ from C .

R-Rule 7. If there are two clauses $z_1 z_2 C_1$ and $z_1 \bar{z}_2 C_2$, where literal $\bar{z}_2$ appears in no other clauses, then remove z_1 from clause $z_1 z_2 C_1$.

Lemma 1. Let $\mathcal{F}$ be a CNF-formula and $\mathcal{F}'$ be the resulting formula after applying R-Rule 7 on $\mathcal{F}$. Then $\mathcal{F}$ is satisfiable if and only if $\mathcal{F}'$ is satisfiable.

Proof. Let $\mathcal{F}$ be the original formula and $\mathcal{F}'$ be the formula after replacing clause $z_1 z_2 C_1$ with clause $z_2 C_1$ in $\mathcal{F}$. Clearly, if $\mathcal{F}'$ is satisfied, then $\mathcal{F}$ is satisfied. We consider the other direction.

Assume that $\mathcal{F}$ is satisfied by an assignment A . We show that $\mathcal{F}'$ is also satisfied. If $z_2 C_1$ is satisfied by assignment A , then $\mathcal{F}'$ is satisfied by assignment A . Next, we assume assignment A satisfies $z_1 z_2 C_1$ but not $z_2 C_1$. Then in A , we have that $z_1 = 1$ and $z_2 = 0$. Since $\bar{z}_2$ is a $(1, 1^+)$ -literal and $z_1 \bar{z}_2 C_2$ is the unique clause containing literal $\bar{z}_2$, we know that all clauses will be satisfied if we replace $z_2 = 0$ with $z_2 = 1$ in A . Thus, $\mathcal{F}'$ is still satisfied. $\square$

R-Rule 8 ([18]). If there is a 2-clause $z_1 z_2$ and a clause $\bar{z}_1 \bar{z}_2 C$ such that literal $\bar{z}_1$ appears in no other clauses, then remove clause $z_1 z_2$.

R-Rule 9 ([18]). If there is a 2-clause $z_1 z_2$ such that either literal z_1 appears only in this clause or there is another 2-clause $\bar{z}_1 \bar{z}_2$, then replace z_1 with $\bar{z}_2$ in $\mathcal{F}$ and then apply R-Rule 3 as often as possible.

R-Rule 10 ([18]). If there are two clauses CD_1 and CD_2 such that $|D_1|, |D_2| \geq 1$ and $|C| \geq 2$, then remove CD_1 and CD_2 from $\mathcal{F}$, and add three new clauses xC , $\bar{x}D_1$, and $\bar{x}D_2$, where x is a new 3-variable.

R-Rule 10 is like the Davis-Putnam resolution in reverse, and thus it is correct.

Definition 1 (Reduced formulas). A CNF-formula is *reduced* if none of the above reduction rules can be applied on it.

Our algorithm will first iteratively apply the above reduction rules to get a reduced formula. We will use $R(\mathcal{F})$ to denote the resulting reduced formula obtained from $\mathcal{F}$. Next, we show some properties of reduced formulas.

Lemma 2. In a reduced CNF-formula $\mathcal{F}$, all variables are 3^+ -variables.

Proof. If there is a 1-variable, then R-Rule 4 could be applied. For a 2-variable in $\mathcal{F}$, if it is a $(2, 0)$ -variable or $(0, 2)$ -variable, then R-Rule 4 could be applied; if it is a $(1, 1)$ -variable, R-Rule 5 could be applied. $\square$

Lemma 3. In a reduced CNF-formula $\mathcal{F}$, if there is a 2-clause xy , then no other clause in $\mathcal{F}$ contains xy , $\bar{x}y$, or $x\bar{y}$.

Proof. If there is a clause containing xy , then R-Rule 2 could be applied. If there is a clause containing $\bar{x}y$ or $x\bar{y}$, then R-Rule 6 could be applied. $\square$

Lemma 4. In a reduced CNF-formula $\mathcal{F}$, if there is a clause xyC , then

- (i) no other clause contains xy ;
- (ii) no other clause contains $\bar{x}y$ or $x\bar{y}$ if x is a 3-variable.

Proof. (i): If there is a clause containing xy , then either R-Rule 2 or R-Rule 10 can be applied.

(ii): Since x is a 3-variable and all $(1^+, 0)$ -literals are reduced by R-Rule 4, x is actually a $(1, 2)$ -variable or $(2, 1)$ -variable. If there is a clause containing $\bar{x}y$, R-Rule 7 would be applicable. If there is a clause containing $x\bar{y}$, then there is only one non-trivial resolvent after resolving on x , which means R-Rule 5 would be applicable. $\square$

Lemma 5. In a reduced CNF-formula $\mathcal{F}$, if there is $(1, 2^+)$ -literal x (let xC be the only clause containing x), then

- (i) $|C| \geq 2$;
- (ii) $\text{var}(C) \cap \text{var}(N^{(2)}(\bar{x}, F)) = \emptyset$, that is, if $y \in N^{(2)}(\bar{x}, F)$, then $y, \bar{y} \notin C$.

Proof. (i): If $|C| = 0$, then R-Rule 4 could be applied; if $|C| = 1$, which means xC is a 2-clause, then R-Rule 9 could be applied since x is a $(1, 2^+)$ -literal.

(ii): Assume that y is a literal in $N^{(2)}(\bar{x}, F)$, in other words, there is a 2-clause $\bar{x}y$. Note that x is a $(1, 2^+)$ -literal. If $y \in C$, then R-Rule 7 could be applied; if $\bar{y} \in C$, then R-Rule 8 could be applied. $\square$

4.2. Branching rules and the algorithm

After getting a reduced formula, we will search for a solution by branching. In a branching operation, we will generate two smaller CNF-formulas such that the original formula is satisfiable if and only if at least one of the two new formulas is satisfiable. The two smaller formulas are generated by specifying the value of a set of literals in the original formula.

The simplest branching rule is that we pick up a variable or literal x from $\mathcal{F}$ and branch into two branches $\mathcal{F}_{x=1}$ and $\mathcal{F}_{x=0}$, where $\mathcal{F}_{x=1}$ and $\mathcal{F}_{x=0}$ are the formulas after assigning $x = 1$ and $x = 0$ in $\mathcal{F}$, respectively. When the picked literal x is a $(1, 1^+)$ -literal, we will apply a stronger branching. Assume that xC is the only clause containing x . Then we branch into two branches $\mathcal{F}_{x=1 \ \& \ C=0}$ and $\mathcal{F}_{x=0}$, where $\mathcal{F}_{x=1 \ \& \ C=0}$ is the resulting formula after assigning 1 to x and 0 to all literals in C in $\mathcal{F}$. The correctness of this branching operation is also easy to observe. Only when all literals in C are assigned 0 do we need to assign 1 to x . Generally, we always pick a variable or literal with the maximum degree in the formula to branch.

The main steps of our algorithm are given in Algorithm 1. The algorithm will execute one step only when all previous steps can not be applied. In Step 2, the algorithm first reduces the formula by applying the reduction rules. Afterwards, in Steps 3-15, the algorithm deals with variables from high-degree to low-degree by branching. In Step 3, the algorithm branches on variables with degree ≥ 6 . Steps 4-13 deal with 5-variables. Steps 14-15 deal with 4-variables. If the maximum degree of the formula is 3, then we apply the algorithm by Wahlström [32], which is Step 16 of our algorithm.

Before analyzing the algorithm, we compare our algorithm with the previous algorithm by Chen and Liu [18]. We can see that they used a simple and uniform branching rule to deal with variables of degree at least 5 and used careful and complicated branching rules for 4-variables. Their bottlenecks contain one case of branching on 5-variables and one case

Algorithm 1: $\text{SAT}(\mathcal{F})$.**Input:** a CNF-formula $\mathcal{F}$ **Output:** 1 or 0 to indicate the satisfiability of $\mathcal{F}$ **Step 1.** If $\mathcal{F} = \emptyset$, return 1. Else if $\mathcal{F}$ contains an empty clause, return 0.**Step 2.** If $\mathcal{F}$ is not a reduced CNF-formula, iteratively apply the reduction rules to reduce it.**Step 3.** If the degree of $\mathcal{F}$ is at least 6, select a variable x with the maximum degree and return $\text{SAT}(\mathcal{F}_{x=1}) \vee \text{SAT}(\mathcal{F}_{x=0})$.**Step 4.** If there is a $(1, 4)$ -literal x (assume that $x\bar{C}$ is the unique clause containing x), return $\text{SAT}(\mathcal{F}_{x=1} \& C=0) \vee \text{SAT}(\mathcal{F}_{x=0})$.**Step 5.** If there is a 5-literal x such that at least two 2-clauses contain x or $\bar{x}$, return $\text{SAT}(\mathcal{F}_{x=1}) \vee \text{SAT}(\mathcal{F}_{x=0})$.**Step 6.** If there are two 5-literals x and y contained in one 2-clause xy , return $\text{SAT}(\mathcal{F}_{x=1}) \vee \text{SAT}(\mathcal{F}_{x=0})$.**Step 7.** If there is a 5-literal x contained in a 2-clause, return $\text{SAT}(\mathcal{F}_{x=1}) \vee \text{SAT}(\mathcal{F}_{x=0})$.**Step 8.** If there is a 5-literal x such that $N(x, \mathcal{F})$ and $N(\bar{x}, \mathcal{F})$ contain at least two 4^- -literals, return $\text{SAT}(\mathcal{F}_{x=1}) \vee \text{SAT}(\mathcal{F}_{x=0})$.**Note:** If there are still some 5-literals, they must be $(2,3)/(3,2)$ -literals. In the next two steps, we let x be a $(2, 3)$ -literal and $x\bar{C}_1, x\bar{C}_2, \bar{x}D_1, \bar{x}D_2$, and $\bar{x}D_3$ be the five clauses containing x or $\bar{x}$.**Step 9.** If there exist 5-literals y_1 and y_2 such that $y_1 \in C_1, y_1 \in D_1, y_2 \in C_2$ and y_2 or $\bar{y}_2 \in D_2$, return $\text{SAT}(\mathcal{F}_{y_1=1}) \vee \text{SAT}(\mathcal{F}_{y_1=0})$.**Step 10.** If there exist 5-literals y_1 and y_2 such that $y_1 \in C_1, \bar{y}_1 \in D_1, y_2 \in C_2$, and $\bar{y}_2 \in D_2$, pick a 5-literal $z \in D_3$ and return $\text{SAT}(R_5(\mathcal{F}_{z=1})) \vee \text{SAT}(\mathcal{F}_{z=0})$, where $R_5(\mathcal{F}_{z=1})$ is the resulting formula after applying R-Rule 5 once on $\mathcal{F}_{z=1}$.**Step 11.** If there is a 5-literal x contained in at least one 4^+ -clause, return $\text{SAT}(\mathcal{F}_{x=1}) \vee \text{SAT}(\mathcal{F}_{x=0})$.**Step 12.** If there is a clause containing both a 5-literal x and a 4^- -literal, return $\text{SAT}(\mathcal{F}_{x=1}) \vee \text{SAT}(\mathcal{F}_{x=0})$.**Step 13.** If there are still some 5-literals, then $\mathcal{F} = \mathcal{F}_5 \wedge \mathcal{F}_{\leq 4}$, where $\mathcal{F}_5$ is a 3-CNF containing only 5-literals and $\mathcal{F}_{\leq 4}$ contains only $3/4$ -literals. We solve $\mathcal{F}_5$ by using the 3-SAT algorithm by Liu [6] (let $A(\mathcal{F}_5)$ denote the result) and return $A(\mathcal{F}_5) \wedge \text{SAT}(\mathcal{F}_{\leq 4})$.**Step 14.** If there is a $(1, 3)$ -literal x (assume that $x\bar{C}$ is the unique clause containing x), return $\text{SAT}(\mathcal{F}_{x=1} \& C=0) \vee \text{SAT}(\mathcal{F}_{x=0})$.**Step 15.** If there is a $(2, 2)$ -literal x , return $\text{SAT}(\mathcal{F}_{x=1}) \vee \text{SAT}(\mathcal{F}_{x=0})$.**Step 16.** Apply the algorithm by Wahlström [32] to solve the instance.

of dealing with 4-variables. We carefully design and analyze the branching rules for 5-variables to avoid one previous bottleneck and also simplify the branching rules for 4-variables. We use Steps 4-13 to deal with different structures of 5-variables, while reference [18] just used a single step like our Step 3 to deal with 5-variables. Steps 14-15 are simple branching rules to deal with 4-variables, while reference [18] used complicated rules to deal with 4-variables. To get further improvements, we may also check some special structures and propose branching rules to deal with them so that we can prove the worst case in our algorithm would not always happen.

We analyze the correctness of the algorithm. Step 2 applies reduction rules, and the correctness has been proven previously. Step 3 deals with variables with degree ≥ 6 . Steps 4-13 deal with 5-variables by simple branching. After Step 7, all clauses containing 5-literals are 3^+ -clause. After Step 8, there is at most one 4^- -literal in the neighbor set of x or $\bar{x}$. In Step 10, there must be a 5-literal in D_3 since $|D_3| \geq 2$, and there is at most one 4^- -literal in it. Thus, the condition of Step 10 holds. If Steps 1-12 do not apply, then $\mathcal{F}$ can be written as $\mathcal{F} = \mathcal{F}_5 \wedge \mathcal{F}_{\leq 4}$, where $\mathcal{F}_5$ is a 3-CNF with $\text{var}(\mathcal{F}_5)$ be the set of 5-variables in $\mathcal{F}$ and $\text{var}(\mathcal{F}_5) \cap \text{var}(\mathcal{F}_{\leq 4}) = \emptyset$. So we can do Step 13. Steps 14-15 deal with 4-variables by branching. When the algorithm comes to the last step, all variables must have a degree of 3, and we apply the algorithm by Wahlström [32] to deal with this special case. The hard part is to analyze the running time bound, which will be presented below.

5. The analysis framework

We use the measure-and-conquer method to analyze the running time bound of our algorithm and adopt $\mu(\mathcal{F})$ defined in (1) as the measure to construct recurrence relations for our branching operations. Before analyzing each detailed step of the algorithm, we first introduce some general frameworks of our analysis and prove some lemmas that will be used as lower bounds in the detailed step analyses.

In each sub-branch of a branching operation, we assign value 1 or 0 to some literals and remove some clauses and literals. If we assign value 1 to a literal x in the formula $\mathcal{F}$, then we will remove all clauses containing x from the formula since all those clauses are satisfied. We also remove all $\bar{x}$ literals from the clauses containing $\bar{x}$ since those literals get value 0. The assignment and removing together are called an *assignment operation*. We may assign values to more than one literal, and we do the assignment operation for each literal.

Let S be a subset of literals. We use $\mathcal{F}_{S=1}$ to denote the resulting formula after assigning 1 to each literal in S and doing assignment operations for literals in S . Note that $\mathcal{F}_{S=1}$ may not be a reduced formula, and we will apply our reduction rules to reduce it. We use $\mathcal{F}'_{S=1}$ to denote the reduced formula obtained from $\mathcal{F}_{S=1}$, i.e., $\mathcal{F}'_{S=1} = R(\mathcal{F}_{S=1})$, and use $\mathcal{F}^*_{S=1}$ to denote the first formula such that R-Rules 1-4 are not applicable during we apply reduction rules on $\mathcal{F}_{S=1}$. We analyze how much we can reduce the measure in each sub-branch by establishing some lower bounds for

$$\Delta_S^* = \mu(\mathcal{F}) - \mu(\mathcal{F}_{S=1}^*);$$

$$\Delta_S = \mu(\mathcal{F}) - \mu(\mathcal{F}'_{S=1}).$$

For the sake of presentation, we define

$$\xi_S^{(1)} = \mu(\mathcal{F}) - \mu(\mathcal{F}_{S=1});$$

$$\begin{aligned}\xi_S^{(2)} &= \mu(\mathcal{F}_{S=1}) - \mu(\mathcal{F}_{S=1}^*); \\ \xi_S^{(3)} &= \mu(\mathcal{F}_{S=1}^*) - \mu(\mathcal{F}_{S=1}'^*).\end{aligned}$$

Thus, it holds that

$$\Delta_S^* = \mu(\mathcal{F}) - \mu(\mathcal{F}_{S=1}^*) = \mu(\mathcal{F}) - \mu(\mathcal{F}_{S=1}) + \mu(\mathcal{F}_{S=1}) - \mu(\mathcal{F}_{S=1}^*) = \xi_S^{(1)} + \xi_S^{(2)}$$

and

$$\Delta_S = \mu(\mathcal{F}) - \mu(\mathcal{F}_{S=1}') = \mu(\mathcal{F}) - \mu(\mathcal{F}_{S=1}^*) + \mu(\mathcal{F}_{S=1}^*) - \mu(\mathcal{F}_{S=1}') = \Delta_S^* + \xi_S^{(3)}.$$

In other words, $\xi_S^{(1)}$ is the amount of measure of $\mathcal{F}$ reduced by only doing assignment operations for literals in S ; $\xi_S^{(2)}$ is the amount of measure of $\mathcal{F}_{S=1}$ reduced by the first four reduction rules; $\xi_S^{(3)}$ is the amount of measure of $\mathcal{F}_{S=1}^*$ reduced by iteratively applying the reduction rules until it becomes a reduced formula.

We analyze the running time of our algorithm by analyzing the branching vector/factor generated by each step. In each branching step, we will branch into two sub-branches. Assume that all literals in S_1 are assigned value 1 in the first sub-branch, and all literals in S_2 are assigned value 1 in the second sub-branch. In the analysis, we will frequently use the following property.

Lemma 6. *It holds that*

- (i) *if we can show that $\Delta_{S_1} \geq p$ and $\Delta_{S_2} \geq q$, then the branching vector generated by this branching rule is covered by $[p, q]$;*
- (ii) *if we can show $\min(\Delta_{S_1}, \Delta_{S_2}) \geq a$ and $\Delta_{S_1} + \Delta_{S_2} \geq b$, then the branching vector generated by this branching rule is covered by $[a, b - a]$.*

Next, we will analyze some lower bounds for Δ_{S_1} , Δ_{S_2} and $\Delta_{S_1} + \Delta_{S_2}$.

According to the assignment operation, we know that all variables of the literals in S will not appear in $\mathcal{F}_{S=1}$. So we have a trivial bound

$$\xi_S^{(1)} \geq \sum_{v \in S} w_{deg(v)}.$$

To get better bounds, we first define some notations. Recall that in a formula $\mathcal{F}$, a literal x is called a neighbor of a literal z if there is a clause containing both z and x . We use $N(z, \mathcal{F})$ to denote the set of neighbors of a literal z in a formula $\mathcal{F}$ and $N^{(k)}(x, \mathcal{F})$ (resp., $N^{(k+)}(z, \mathcal{F})$) to denote the neighbors of z in k -clauses (resp., k^+ -clauses) in $\mathcal{F}$.

Definition 2. For a literal x in a reduced formula $\mathcal{F}$ and $i \in \mathbb{N}^+$, we define the following notations:

- $n_i(x)$: the number of literals with degree i that appear in $N(x, \mathcal{F})$, i.e., $n_i(x) = |\{y : y \in N(x, \mathcal{F}) \text{ and } deg(y) = i\}|$;
- $n_i'(x)$: the number of literals with degree i that appear in $N^{(2)}(x, \mathcal{F})$, i.e., $n_i'(x) = |\{y : y \in N^{(2)}(x, \mathcal{F}) \text{ and } deg(y) = i\}|$;
- $n_i''(x)$: the number of literals with degree i that appear in $N^{(3+)}(x, \mathcal{F})$, i.e., $n_i''(x) = |\{y : y \in N^{(3+)}(x, \mathcal{F}) \text{ and } deg(y) = i\}|$;
- $t_{i,1}(x)$: the number of i -variables y such that only one of y and $\bar{y}$ appears in $N(x, \mathcal{F})$, i.e., $t_{i,1}(x) = |\{var(y) : |\{y, \bar{y}\} \cap N(x, \mathcal{F})| = 1 \text{ and } deg(y) = i\}|$;
- $t_{i,2}(x)$: the number of i -variables y such that both of y and $\bar{y}$ appear in $N(x, \mathcal{F})$, i.e., $t_{i,2}(x) = |\{var(y) : |\{y, \bar{y}\} \cap N(x, \mathcal{F})| = 2 \text{ and } deg(y) = i\}|$.

Example. Let $\mathcal{F} = (xy_1z_1) \wedge (xy_2\bar{z}_1l_1) \wedge (xy_3\bar{z}_3) \wedge (\bar{x}z_2z_3\bar{l}_2) \wedge (\bar{x}z_4) \wedge \mathcal{F}_1$, where x, z_1, z_2, z_3, z_4 are 5-variables, y_1, y_2, y_3 are 4-variables, and l_1, l_2 are 3-variables. Then, in this case, we have:

- $n_3(x) = n_3''(x) = t_{3,1}(x) = |\{l_1\}| = 1$;
- $n_4(x) = n_4''(x) = t_{4,1}(x) = |\{y_1, y_2, y_3\}| = 3$;
- $n_5(x) = n_5''(x) = |\{z_1, \bar{z}_1, z_3\}| = 3$;
- $t_{5,1}(x) = |\{z_3\}| = 1$ and $t_{5,2}(x) = |\{z_1\}| = 1$;
- $n_5(\bar{x}) = |\{z_2, z_3, z_4\}| = 3$, $n_5'(\bar{x}) = |\{z_4\}| = 1$, and $n_5''(\bar{x}) = |\{z_2, z_3\}| = 2$.

In a reduced formula $\mathcal{F}$, there is no 1-clause since R-Rule 4 is not applicable. So for a literal x , it holds that

$$n_i(x) = n_i'(x) + n_i''(x).$$

By Lemma 4, for a literal x , we know that all literals in $N(x, \mathcal{F})$ are different from each other. So for any variable y , there is at most one literal y and at most one literal $\bar{y}$ appearing in $N(x, \mathcal{F})$, which implies that

$$n_i(x) = t_{i,1}(x) + 2t_{i,2}(x).$$

Next, we give some lower bounds on $\xi_S^{(1)}$, $\xi_S^{(2)}$, and $\Delta_{S_1}^* + \Delta_{S_2}^*$, which will be used to prove our main results. The following lemma shows how much the measure decreases after we do the assignment operation to a single literal.

Lemma 7. Assume that $\mathcal{F}$ is a reduced CNF-formula with degree d . Let $S = \{x\}$, where x is a literal with degree d in $\mathcal{F}$. It holds that

$$\xi_S^{(1)} \geq w_d + \sum_{3 \leq i \leq d} n_i(x) \delta_i.$$

Proof. After assigning value 1 to literal x , all clauses containing x will be removed from $\mathcal{F}$, that is, all literals in $N(x, \mathcal{F})$ will be removed from $\mathcal{F}$. By Lemma 4, all literals in $N(x, \mathcal{F})$ are different from each other. For a variable y , there are at most two literals of it (y and $\bar{y}$) in $N(x, \mathcal{F})$. So the degree of variable y will decrease at most 2. Moreover, if y is a 3-variable, by Lemma 4, we know that y and $\bar{y}$ will not simultaneously appear in $N(x, \mathcal{F})$ since there are no other clauses containing $x\bar{y}$ (i.e., $t_{3,2}(x) = 0$). So the degree of y will decrease by at most 1 if y is a 3-variable.

Note that $\xi_S^{(1)}$ expresses how much the measure of $\mathcal{F}$ decreases after the assignment operation. So it holds that

$$\xi_S^{(1)} \geq w_d + \sum_{3 \leq i \leq d} t_{i,1}(x) \delta_i + \sum_{4 \leq i \leq d} t_{i,2}(x) (\delta_i + \delta_{i-1}).$$

Since $t_{3,2}(x) = 0$, can write it as

$$\xi_S^{(1)} = w_d + \sum_{3 \leq i \leq d} t_{i,1}(x) \delta_i + \sum_{3 \leq i \leq d} t_{i,2}(x) (\delta_i + \delta_{i-1}).$$

By $\delta_{i-1} \geq \delta_i$ and $\delta_i \geq \delta_d$ for $3 \leq i \leq d$, we have

$$\begin{aligned} \xi_S^{(1)} &\geq w_d + \sum_{3 \leq i \leq d} t_{i,1}(x) \delta_i + \sum_{3 \leq i \leq d} t_{i,2}(x) (\delta_i + \delta_i) \\ &= w_d + \sum_{3 \leq i \leq d} (t_{i,1}(x) + 2t_{i,2}(x)) \delta_i. \end{aligned}$$

Note that $n_i(x) = t_{i,1}(x) + 2t_{i,2}(x)$ holds for $3 \leq i \leq d$. We finally obtain

$$\xi_S^{(1)} \geq w_d + \sum_{3 \leq i \leq d} n_i(x) \delta_i. \quad \square$$

Lemma 8. Assume that $\mathcal{F}$ is a reduced CNF-formula with degree d . Let $S = \{x\}$, where x is a $(j, d - j)$ -literal in $\mathcal{F}$. It holds that

$$\xi_S^{(1)} \geq w_d + j\delta_d.$$

Proof. By Lemma 7 and $\delta_i \geq \delta_d$ for $3 \leq i \leq d$, we get

$$\xi_S^{(1)} \geq w_{deg(x)} + \sum_{3 \leq i \leq d} n_i(x) \delta_i \geq w_d + \sum_{3 \leq i \leq d} n_i(x) \delta_d.$$

As x is a $(j, d - j)$ -literal, there are j clauses containing literal x . Since there is no 1-clause in $\mathcal{F}$ (all 1-clauses are reduced by R-Rule 4), literal x has at least j neighbors, i.e., $\sum_{3 \leq i \leq d} n_i(x) \geq j$. We further obtain

$$\xi_S^{(1)} \geq w_d + j\delta_d. \quad \square$$

Lemma 9. Assume that $\mathcal{F}$ is a reduced CNF-formula with degree d . Let $S = \{x\}$, where x is a literal in $\mathcal{F}$. It holds that

$$\xi_S^{(2)} \geq n'_3(\bar{x}) w_3 + \sum_{4 \leq i \leq d} n'_i(\bar{x}) w_{i-1}.$$

Proof. Recall that $\xi_S^{(2)}$ expresses how much the measure decreases after applying reduction Rule 1-4 on $\mathcal{F}_{S=1}$. All literals in $N^{(2)}(\bar{x}, \mathcal{F})$ will be assigned value 1 since they will be contained in 1-clauses in $\mathcal{F}_{S=1}$ and R-Rule 4 is applied.

For a literal $y \in N^{(2)}(\bar{x}, \mathcal{F})$, we know that there are no other clauses containing xy (i.e., $y \notin N(x, \mathcal{F})$) by Lemma 3. So if $var(y)$ is an i -variable in $\mathcal{F}$, then the degree of y in $\mathcal{F}_{S=1}$ is at least $i - 1$. Moreover, for a literal $y \in N^{(2)}(x, \mathcal{F})$ such that $var(y)$ is a 3-variable, by Lemma 4 we know that there are no other clauses containing xy or $x\bar{y}$ (i.e., $y, \bar{y} \notin N(x, \mathcal{F})$). So the degree of y would still be 3 in $\mathcal{F}_{S=1}$.

Since all variables corresponding to the literals in $N^{(2)}(\bar{x}, \mathcal{F})$ would not appear in $\mathcal{F}'_{S=1}$ and these variables are different from each other by Lemma 3, we obtain

$$\xi_S^{(2)} \geq n'_3(\bar{x})w_3 + \sum_{4 \leq i \leq d} n'_i(\bar{x})w_{i-1}. \quad \square$$

After getting the lower bounds on $\xi_S^{(1)}$ and $\xi_S^{(2)}$, we are going to establish the lower bound for $\Delta_{S_1}^* + \Delta_{S_2}^*$ for the case that $S_1 = \{x\}$ and $S_2 = \{\bar{x}\}$.

Lemma 10. Assume that $\mathcal{F}$ is a reduced CNF-formula of degree d . Let $S_1 = \{x\}$ and $S_2 = \{\bar{x}\}$, where x is a d -variable in $\mathcal{F}$. It holds that

$$\Delta_{S_1}^* + \Delta_{S_2}^* \geq 2w_d + 2d\delta_d + (n'_3(x) + n'_3(\bar{x}))(2w_3 - 2\delta_d) + \sum_{4 \leq i \leq d} (n'_i(x) + n'_i(\bar{x}))(w_i - 2\delta_d).$$

Proof. By Lemma 7 and Lemma 9, we have

$$\xi_{S_1}^{(1)} \geq w_d + \sum_{3 \leq i \leq d} n_i(x)\delta_i \quad \text{and} \quad \xi_{S_1}^{(2)} \geq n'_3(\bar{x})w_3 + \sum_{4 \leq i \leq d} n'_i(\bar{x})w_{i-1}.$$

We can get lower bounds of $\xi_{S_2}^{(1)}$ and $\xi_{S_2}^{(2)}$ by the same way, and then we have:

$$\begin{aligned} \Delta_{S_1}^* &= \xi_{S_1}^{(1)} + \xi_{S_1}^{(2)} \geq w_d + \sum_{3 \leq i \leq d} n_i(x)\delta_i + n'_3(\bar{x})w_3 + \sum_{4 \leq i \leq d} n'_i(\bar{x})w_{i-1}; \\ \Delta_{S_2}^* &= \xi_{S_2}^{(1)} + \xi_{S_2}^{(2)} \geq w_d + \sum_{3 \leq i \leq d} n_i(\bar{x})\delta_i + n'_3(x)w_3 + \sum_{4 \leq i \leq d} n'_i(x)w_{i-1}. \end{aligned} \quad (7)$$

As $\sum_{3 \leq i \leq d} (n'_i(x) + n'_i(\bar{x}))$ is the number of 2-clauses containing x or $\bar{x}$ and there are d clauses containing x or $\bar{x}$, we know that the number of 3^+ -clauses that contains x or $\bar{x}$ is $d - \sum_{3 \leq i \leq d} (n'_i(x) + n'_i(\bar{x}))$. Recall that $n''_i(x)$ is the number of literals with degree i that appear in $N^{(3+)}(x, \mathcal{F})$. We get

$$\sum_{3 \leq i \leq d} (n''_i(x) + n''_i(\bar{x})) \geq 2(d - \sum_{3 \leq i \leq d} (n'_i(x) + n'_i(\bar{x}))). \quad (8)$$

By (7) and summing $\Delta_{S_1}^*$ and $\Delta_{S_2}^*$ up, we have

$$\begin{aligned} \Delta_{S_1}^* + \Delta_{S_2}^* &\geq (w_d + \sum_{3 \leq i \leq d} n_i(x)\delta_i + n'_3(\bar{x})w_3 + \sum_{4 \leq i \leq d} n'_i(\bar{x})w_{i-1}) + (w_d + \sum_{3 \leq i \leq d} n_i(\bar{x})\delta_i + n'_3(x)w_3 + \sum_{4 \leq i \leq d} n'_i(x)w_{i-1}) \\ &= 2w_d + (n'_3(x) + n'_3(\bar{x}))w_3 + \sum_{4 \leq i \leq d} (n'_i(x) + n'_i(\bar{x}))w_{i-1} + \sum_{3 \leq i \leq d} (n_i(x) + n_i(\bar{x}))\delta_i. \end{aligned}$$

Next, we first consider the lower bound on the term $\sum_{3 \leq i \leq d} (n_i(x) + n_i(\bar{x}))\delta_i$ and then further analyze $\Delta_{S_1}^* + \Delta_{S_2}^*$.

By $\delta_i \geq \delta_d$ for $3 \leq i \leq d$ and (8), we have

$$\begin{aligned} \sum_{3 \leq i \leq d} (n''_i(x) + n''_i(\bar{x}))\delta_i &\geq \sum_{3 \leq i \leq d} (n''_i(x) + n''_i(\bar{x}))\delta_d \\ &\geq 2(d - \sum_{3 \leq i \leq d} (n'_i(x) + n'_i(\bar{x})))\delta_d. \end{aligned}$$

With $n_i(x) = n'_i(x) + n''_i(x)$ and $n_i(\bar{x}) = n'_i(\bar{x}) + n''_i(\bar{x})$, we get

$$\begin{aligned} \sum_{3 \leq i \leq d} (n_i(x) + n_i(\bar{x}))\delta_i &= \sum_{3 \leq i \leq d} (n'_i(x) + n'_i(\bar{x}))\delta_i + \sum_{3 \leq i \leq d} (n''_i(x) + n''_i(\bar{x}))\delta_i \\ &\geq \sum_{3 \leq i \leq d} (n'_i(x) + n'_i(\bar{x}))\delta_i + 2(d - \sum_{3 \leq i \leq d} (n'_i(x) + n'_i(\bar{x})))\delta_d \\ &\geq 2d\delta_d + \sum_{3 \leq i \leq d} (n'_i(x) + n'_i(\bar{x}))(\delta_i - 2\delta_d). \end{aligned}$$

Next we continue the previous analysis on $\Delta_{S_1}^* + \Delta_{S_2}^*$. By applying the above result, we have

$$\begin{aligned}
\Delta_{S_1}^* + \Delta_{S_2}^* &\geq 2w_d + (n'_3(x) + n'_3(\bar{x}))w_3 + \sum_{4 \leq i \leq d} (n'_i(x) + n'_i(\bar{x}))w_{i-1} \\
&\quad + \sum_{3 \leq i \leq d} (n_i(x) + n_i(\bar{x}))\delta_i \\
&= 2w_d + (n'_3(x) + n'_3(\bar{x}))w_3 + \sum_{4 \leq i \leq d} (n'_i(x) + n'_i(\bar{x}))w_{i-1} \\
&\quad + 2d\delta_d + \sum_{3 \leq i \leq d} (n'_i(x) + n'_i(\bar{x}))(\delta_i - 2\delta_d).
\end{aligned}$$

Expanding $3 \leq i \leq d$ to $i = 3$ and $4 \leq i \leq d$ in the last term and then combining like terms, we get

$$\begin{aligned}
\Delta_{S_1}^* + \Delta_{S_2}^* &= 2w_d + 2d\delta_d + (n'_3(x) + n'_3(\bar{x}))w_3 + \sum_{4 \leq i \leq d} (n'_i(x) + n'_i(\bar{x}))w_{i-1} \\
&\quad + (n'_3(x) + n'_3(\bar{x}))(\delta_3 - 2\delta_d) + \sum_{4 \leq i \leq d} (n'_i(x) + n'_i(\bar{x}))(\delta_i - 2\delta_d) \\
&\geq 2w_d + 2d\delta_d + (n'_3(x) + n'_3(\bar{x}))(w_3 + \delta_3 - 2\delta_d) + \sum_{4 \leq i \leq d} (n'_i(x) + n'_i(\bar{x}))(w_{i-1} + \delta_i - 2\delta_d) \\
&= 2w_d + 2d\delta_d + (n'_3(x) + n'_3(\bar{x}))(2w_3 - 2\delta_d) + \sum_{4 \leq i \leq d} (n'_i(x) + n'_i(\bar{x}))(w_i - 2\delta_d). \quad \square
\end{aligned}$$

In our algorithm, we apply a stronger branching for a $(1, 4)/(1, 3)$ -literal x . Assume xC is the unique clause containing literal x . The following lemma shows the lower bounds on $\Delta_{S_1} + \Delta_{S_2}$ and $\min(\Delta_{S_1}, \Delta_{S_2})$ for the case $S_1 = \{x\} \cup \bar{C}$ and $S_2 = \{\bar{x}\}$.

Lemma 11. Assume that $\mathcal{F}$ is a reduced CNF-formula of degree $d = 4$ or 5 . Let x be a $(1, d-1)$ -literal and xC be the unique clause containing x in $\mathcal{F}$. Let $S_1 = \{x\} \cup \bar{C}$ and $S_2 = \{\bar{x}\}$. It holds that

$$\Delta_{S_1} + \Delta_{S_2} \geq 2w_d + 3w_3 + (2d-3)\delta_d \text{ and } \min(\Delta_{S_1}, \Delta_{S_2}) \geq w_d + 2w_3.$$

Proof. We first consider Δ_{S_1} . By Lemma 5, we know that $|C| \geq 2$ and $\text{var}(C) \cap \text{var}(N^{(2)}(\bar{x}, \mathcal{F})) = \emptyset$. So, after assigning value 1 to all literals in S_1 , all 2-clauses containing $\bar{x}$ in $\mathcal{F}$ would become 1-clauses in $\mathcal{F}_{S_1=1}$ and the degree of each variable in these 1-clauses is the same as that in $\mathcal{F}$. Then, by applying R-Rule 4, all variables in these 1-clauses get assignments. Thus, all variables in $\text{var}(C) \cup \text{var}(N^{(2)}(\bar{x}, \mathcal{F}))$ would not appear in $\mathcal{F}'_{S_1=1}$. Since all variables in C are 3^+ -variables and $w_3 \leq w_i$ for $i \geq 3$, we preliminarily have

$$\begin{aligned}
\Delta_{S_1} &\geq \sum_{v \in S_1} w_{\deg(v)} + \sum_{3 \leq i \leq d} n'_i(\bar{x})w_i \\
&\geq w_d + |C|w_3 + \sum_{3 \leq i \leq d} n'_i(\bar{x})w_i.
\end{aligned} \tag{9}$$

Note that when we assign 1 to each literal in $S_1 = \{x\} \cup \bar{C}$, for a literal $y \in \bar{C}$, the neighbors of y would also be removed from the formula, which may further decrease the measure. A neighbor of a literal $y \in \bar{C}$, say $z \in N(y, \mathcal{F})$, can further contribute some decrease to the above Δ_{S_1} if it satisfies the following two conditions:

- (i) Literals z and $\bar{z}$ do not appear in $\bar{C}$. Otherwise the decrease of Δ_{S_1} contributed by z is already accounted in the term $|C|w_3$.
- (ii) Literals z and $\bar{z}$ do not appear in $N^{(2)}(\bar{x}, \mathcal{F})$. Otherwise after we assign value 1 to literal x , there will be a 1-clause $\{z\}$ or $\{\bar{z}\}$ and then the variable z will get assignment by applying R-Rule 4. In this case, the decrease of Δ_{S_1} contributed by z is already accounted in the term $\sum_{3 \leq i \leq d} n'_i(\bar{x})w_i$.

So in order to capture some further decrease of Δ_{S_1} , we define

$$N_c = \{\text{var}(z) : z \in \bigcup_{y \in \bar{C}} N(y, \mathcal{F}) \text{ and } z, \bar{z} \notin \bar{C} \cup N^{(2)}(\bar{x}, \mathcal{F})\}.$$

As the degree of each variable in N_c decreases by at least 1 after assigning each literal in S and these decreases are not accounted before, with $\delta_d \leq \delta_i$ for $i \leq d$, we further get

$$\begin{aligned}\Delta_{S_1} &\geq w_d + |C|w_3 + \sum_{3 \leq i \leq d} n'_i(\bar{x})w_i + |N_c|\delta_d \\ &\geq w_d + |C|w_3 + \sum_{3 \leq i \leq d} n'_i(\bar{x})w_3 + |N_c|\delta_d.\end{aligned}$$

Next, we consider Δ_{S_2} . By Lemma 7 and $\delta_d \leq \delta_i$ for $3 \leq i \leq d$, we have

$$\Delta_{S_2} \geq w_d + \sum_{3 \leq i \leq d} n_i(\bar{x})\delta_i \geq w_d + \sum_{3 \leq i \leq d} n_i(\bar{x})\delta_d.$$

Let $p = \sum_{3 \leq i \leq d} n'_i(\bar{x})$ and $q = \sum_{3 \leq i \leq d} n''_i(\bar{x})$. Note that p is also the number of 2-clauses containing $\bar{x}$ and $0 \leq p \leq d-1$. Since there are $d-1$ clauses containing $\bar{x}$, it holds that $q \geq 2(d-1-p)$. Recall that $n_i(\bar{x}) = n'_i(\bar{x}) + n''_i(\bar{x})$ for $i \in N^+$, we have

$$\sum_{3 \leq i \leq d} n_i(\bar{x}) = p + q \geq p + 2(d-1-p) = 2d-2-p.$$

So

$$\Delta_{S_1} \geq w_d + |C|w_3 + pw_3 + |N_c|\delta_d \text{ and } \Delta_{S_2} \geq w_d + (2d-2-p)\delta_d.$$

Summing them up, we get

$$\Delta_{S_1} + \Delta_{S_2} \geq 2w_d + |C|w_3 + (|N_c| + 2d-2)\delta_d + p(w_3 - \delta_d).$$

Since $0 \leq p \leq d-1$, it holds that $\Delta_{S_2} \geq w_d + (d-1)\delta_d$.

Next, let us consider the following two cases.

Case 1. $\bar{x}$ is contained in at least one 2-clause, i.e., $p \geq 1$. With $|C| \geq 2$, we have

$$\Delta_{S_1} + \Delta_{S_2} \geq 2w_d + 2w_3 + (2d-2)\delta_d + w_3 - \delta_d = 2w_d + 3w_3 + (2d-3)\delta_d.$$

For this case, we also have $\Delta_{S_1} \geq w_d + 3w_3$, $\Delta_{S_2} \geq w_d + (d-1)\delta_d$, and $\min(\Delta_{S_1}, \Delta_{S_2}) \geq w_d + \min(3w_3, (d-1)\delta_d) \geq 2w_3$ since that $2w_3 < (d-1)\delta_d$ holds for $d=4$ and $d=5$.

Case 2. All clauses containing $\bar{x}$ are 3^+ -clauses, i.e., $N^{(2)}(\bar{x}, \mathcal{F}) = \emptyset$ and $p=0$. Recall that $|C| \geq 2$. We further consider two sub-cases.

Case 2.1. $|C| \geq 3$. Then we have

$$\Delta_{S_1} + \Delta_{S_2} \geq 2w_d + 3w_3 + (2d-2)\delta_d.$$

For this sub-case, we also have $\Delta_{S_1} \geq w_d + 3w_3$, $\Delta_{S_2} \geq w_d + (d-1)\delta_d$, and $\min(\Delta_{S_1}, \Delta_{S_2}) \geq w_d + \min(3w_3, (d-1)\delta_d) \geq 2w_3$ since that $2w_3 < (d-1)\delta_d$ holds for $d=4$ and $d=5$.

Case 2.2. $|C| = 2$ and assume without loss of generality the unique clause containing literal x is xy_1y_2 . Note that since R-Rule 4 is not applicable, we have $\bigcup_{y \in \bar{C}} N(y, \mathcal{F}) = N(\bar{y}_1, \mathcal{F}) \cup N(\bar{y}_2, \mathcal{F}) \neq \emptyset$. We claim that $|N_c| \geq 1$ and the reason is as follows. Assume $N_c = \emptyset$. Recall that $N_c = \{var(z) : z \in \bigcup_{y \in \bar{C}} N(y, \mathcal{F}) \text{ and } z, \bar{z} \notin \bar{C} \cup N^{(2)}(\bar{x}, \mathcal{F})\}$. With $\bigcup_{y \in \bar{C}} N(y, \mathcal{F}) \neq \emptyset$, $N^{(2)}(\bar{x}, \mathcal{F}) = \emptyset$ and $\bar{C} = \bar{y}_1\bar{y}_2$, we deduce that $N(\bar{y}_1, \mathcal{F}) = \{\bar{y}_2\}$ and $N(\bar{y}_2, \mathcal{F}) = \{\bar{y}_1\}$. Since R-Rule 2 is not applicable, it holds that the 2-clause $\bar{y}_1\bar{y}_2$ is unique, and moreover, both $\bar{y}_1$ and $\bar{y}_2$ are $(1, 1^+)$ -literals. This implies that R-Rule 9 is applicable, which contradicts with that $\mathcal{F}$ is reduced. Thus $N_c \neq \emptyset$ and so $|N_c| \geq 1$ holds. Then we have

$$\begin{aligned}\Delta_{S_1} + \Delta_{S_2} &\geq 2w_d + 2w_3 + (1+2d-2)\delta_d = 2w_d + 2w_3 + (2d-1)\delta_d \\ &\geq 2w_d + 3w_3 + (2d-3)\delta_d.\end{aligned}$$

For this sub-case, we also have $\Delta_{S_1} \geq w_d + 2w_3$, $\Delta_{S_2} \geq w_d + (d-1)\delta_d$, and $\min(\Delta_{S_1}, \Delta_{S_2}) \geq w_d + \min(2w_3, (d-1)\delta_d) \geq 2w_3$ since that $2w_3 < (d-1)\delta_d$ holds for $d=4$ and $d=5$. $\square$

As shown in Algorithm 1, we consider several cases for 5-literals. The following lemma is a corollary based on Lemma 7 in order to get tighter bounds on $\Delta_{S_1} + \Delta_{S_2}$ for the case $S_1 = \{x\}$ and $S_2 = \{\bar{x}\}$ where x is a 5-variable.

Lemma 12. Assume that $\mathcal{F}$ is a reduced CNF-formula of $d=5$. Let $S_1 = \{x\}$ and $S_2 = \{\bar{x}\}$, where x is a $(2, 3)/(3, 2)$ -literal in $\mathcal{F}$. If all clauses containing x or $\bar{x}$ are 3^+ -clauses, it holds that

$$\xi_{S_1}^{(1)} + \xi_{S_2}^{(1)} \geq 2w_5 + \sum_{3 \leq i \leq 5} (n_i(x) + n_i(\bar{x}))\delta_i + (t_{5,2}(x) + t_{5,2}(\bar{x}))(\delta_4 - \delta_5).$$

Proof. From the proof of Lemma 7, we have

$$\begin{aligned}
 \xi_S^{(1)} &\geq w_5 + \sum_{3 \leq i \leq 5} t_{i,1}(x) \delta_i + \sum_{4 \leq i \leq 5} t_{i,2}(x) (\delta_i + \delta_{i-1}) \\
 &= w_5 + \sum_{3 \leq i \leq 5} (n_i(x) - 2t_{i,2}(x)) \delta_i + \sum_{4 \leq i \leq 5} t_{i,2}(x) (\delta_i + \delta_{i-1}) \\
 &= w_5 + \sum_{3 \leq i \leq 5} n_i(x) \delta_i - \sum_{3 \leq i \leq 5} 2t_{i,2}(x) \delta_i + \sum_{4 \leq i \leq 5} t_{i,2}(x) (\delta_{i-1} + \delta_i).
 \end{aligned}$$

As mentioned before, $t_{3,2}(x) = 0$ holds, so we have

$$\begin{aligned}
 \xi_S^{(1)} &\geq w_5 + \sum_{3 \leq i \leq 5} n_i(x) \delta_i - \sum_{4 \leq i \leq 5} 2t_{i,2}(x) \delta_i + \sum_{4 \leq i \leq 5} t_{i,2}(x) (\delta_{i-1} + \delta_i) \\
 &= w_5 + \sum_{3 \leq i \leq 5} n_i(x) \delta_i + \sum_{4 \leq i \leq 5} t_{i,2}(x) (\delta_{i-1} - \delta_i).
 \end{aligned}$$

Since $\delta_{i-1} - \delta_i = 0$ when $i = 4$, we have

$$\xi_{S_1}^{(1)} \geq w_5 + \sum_{3 \leq i \leq 5} n_i(x) \delta_i + t_{5,2}(x) (\delta_4 - \delta_5).$$

Similarly, we can get

$$\xi_{S_2}^{(1)} \geq w_5 + \sum_{3 \leq i \leq 5} n_i(\bar{x}) \delta_i + t_{5,2}(\bar{x}) (\delta_4 - \delta_5).$$

Summing $\xi_{S_1}^{(1)}$ and $\xi_{S_2}^{(1)}$ up, we have

$$\xi_{S_1}^{(1)} + \xi_{S_2}^{(1)} \geq 2w_5 + \sum_{3 \leq i \leq 5} (n_i(x) + n_i(\bar{x})) \delta_i + (t_{5,2}(x) + t_{5,2}(\bar{x})) (\delta_4 - \delta_5). \quad \square$$

The following lemma is a corollary of Lemma 12, which will also be used in our analysis to simplify the arguments.

Lemma 13. Assume that $\mathcal{F}$ is a reduced CNF-formula of $d = 5$. Let $S_1 = \{x\}$ and $S_2 = \{\bar{x}\}$, where x is a $(2, 3)/(3, 2)$ -literal in $\mathcal{F}$. If all clauses containing x or $\bar{x}$ are 3^+ -clauses, $\sum_{3 \leq i \leq 5} (n_i(x) + n_i(\bar{x})) \geq g$, and $\sum_{3 \leq i \leq 4} (n_i(x) + n_i(\bar{x})) \geq h$, then it holds that

$$\xi_{S_1}^{(1)} + \xi_{S_2}^{(1)} \geq 2w_5 + g\delta_5 + h(w_3 - \delta_5) + (t_{5,2}(x) + t_{5,2}(\bar{x})) (\delta_4 - \delta_5).$$

Proof. By Lemma 12 we have

$$\begin{aligned}
 \xi_{S_1}^{(1)} + \xi_{S_2}^{(1)} &\geq 2w_5 + \sum_{3 \leq i \leq 5} (n_i(x) + n_i(\bar{x})) \delta_i + (t_{5,2}(x) + t_{5,2}(\bar{x})) (\delta_4 - \delta_5) \\
 &= 2w_5 + \sum_{3 \leq i \leq 4} (n_i(x) + n_i(\bar{x})) \delta_i + (n_5(x) + n_5(\bar{x})) \delta_5 + (t_{5,2}(x) + t_{5,2}(\bar{x})) (\delta_4 - \delta_5).
 \end{aligned}$$

Let $g' = \sum_{3 \leq i \leq 5} (n_i(x) + n_i(\bar{x})) \geq g$ and $h' = \sum_{3 \leq i \leq 4} (n_i(x) + n_i(\bar{x})) \geq h$. Since $\delta_3 = \delta_4 = w_3$ and $n_5(x) + n_5(\bar{x}) = g' - h'$, we have

$$\begin{aligned}
 \xi_{S_1}^{(1)} + \xi_{S_2}^{(1)} &\geq 2w_5 + h'w_3 + (g' - h')\delta_5 + (t_{5,2}(x) + t_{5,2}(\bar{x})) (\delta_4 - \delta_5) \\
 &= 2w_5 + g'\delta_5 + h'(w_3 - \delta_5) + (t_{5,2}(x) + t_{5,2}(\bar{x})) (\delta_4 - \delta_5).
 \end{aligned}$$

Note that $\delta_5 > 0$ and $w_3 - \delta_5 > 0$, so it holds that

$$\xi_{S_1}^{(1)} + \xi_{S_2}^{(1)} \geq 2w_5 + g\delta_5 + h(w_3 - \delta_5) + (t_{5,2}(x) + t_{5,2}(\bar{x})) (\delta_4 - \delta_5). \quad \square$$

6. Step analysis

Equipped with the above lower bounds, we are ready to analyze the branching vector of each step in the algorithm.

6.1. Step 2

Step 2. If $\mathcal{F}$ is not a reduced CNF-formula, iteratively apply the reduction rules to reduce it.

In this step, we only apply reduction rules to reduce the formula. However, it is still important to show that the measure will never increase when applying reduction rules, and reduction operations use only polynomial time.

Lemma 14. For any CNF-formula $\mathcal{F}$, it holds that $\mu(R(\mathcal{F})) \leq \mu(\mathcal{F})$.

Proof. It suffices to verify that each reduction rule will not increase the measure of the formula.

R-Rules 1-8 simply remove some literals, which will not increase the measure of the formula.

Next, we consider R-Rules 9 and 10. Note that now R-Rules 4 and 5 are not applicable, and then all variables in $\mathcal{F}$ are 3^+ -variables. The reason is below. If there is a 1-variable, we could apply R-Rule 4. For a 2-variable in $\mathcal{F}$, if it is a $(2, 0)$ -variable or $(0, 2)$ -variable, R-Rule 4 would be applicable; if it is a $(1, 1)$ -variable, R-Rule 5 would be applicable.

In R-Rule 9, we replace z_1 with $\bar{z}_2$ and then apply R-Rule 3 as often as possible. Without loss of generality, we assume the degree of z_1 is i and the degree of z_2 is j such that $3 \leq i \leq j$. After applying this rule, the degree of z_2 will become $i + j - 2$ since clause $z_1 z_2$ is removed by R-Rule 3 after the replacing operation. Next, we show that $\mu(R(\mathcal{F})) - \mu(\mathcal{F}) = w_{i+j-2} - w_i - w_j \leq 0$ holds.

By (2), (5) and (3), we know that:

if $i = 3$, then

$$w_{3+j-2} - w_3 - w_j = w_{j+1} - w_j - w_3 = \delta_{j+1} - w_3 \leq \delta_4 - w_3 = 0;$$

if $i = 4$, then

$$w_{4+j-2} - w_4 - w_j = w_{j+2} - w_j - w_4 = \delta_{j+2} + \delta_{j+1} - w_4 \leq \delta_6 + \delta_5 - w_4 < 0;$$

if $i \geq 5$, then

$$w_{i+j-2} - w_i - w_j = (i + j - 2) - i - j = -2 < 0.$$

For R-Rule 10, two clauses CD_1 and CD_2 in $\mathcal{F}$ are replaced with three clauses $x C, \bar{x} D_1$ and $\bar{x} D_2$. We introduce a 3-variable x and also decrease the degree of each literal in C by 1. The introduction of x increases the measure of the formula by w_3 . On the other hand, since all variables are 3^+ -variable in $\mathcal{F}$ and $|C| \geq 2$, the removing of C decreases the measure by at least $2 \min\{\delta_i | i \geq 3\} = 2$. Since $w_3 < 2$, we know that

$$\mu(R(\mathcal{F})) - \mu(\mathcal{F}) = w_3 - 2 \min\{\delta_i | i \geq 3\} = w_3 - 2 < 0. \quad \square$$

Lemma 15. For any CNF-formula $\mathcal{F}$, we can apply the reduction rules in polynomial time to transfer it to $R(\mathcal{F})$.

Proof. Each application of any one of the first eight reduction rules removes some literal from the formula, and then it decreases $L(\mathcal{F})$ by at least 1. For R-Rule 9, after replacing z_1 with $\bar{z}_2$, the clause $z_1 z_2$ will be removed by R-Rule 3. So each application of it decreases $L(\mathcal{F})$ by at least 2.

It is easy to see that each application of R-Rule 10 increases $L(\mathcal{F})$ by at most 1. In the proof of Lemma 14, we have shown that the measure $\mu(\mathcal{F})$ by decreases at least $2 - w_3$ in this step. Since w_3 is a constant less than 2, let $w_3 = 2 - \epsilon$ for a constant ϵ , then $\mu(\mathcal{F})$ decreases by at least ϵ after applying R-Rule 10.

In order to make the proof clear, we define a new measure $M(\mathcal{F}) = L(\mathcal{F}) + 2\mu(\mathcal{F})/\epsilon$. It is easy to see that $M(\mathcal{F})$ is bounded by a polynomial of $L(\mathcal{F})$. For R-Rule 1-9, each decreases $M(\mathcal{F})$ by at least 1. For R-Rule 10, it increases $L(\mathcal{F})$ by at most 1. Assume that the resulting CNF-formula after applying R-Rule 10 is $\mathcal{F}'$. It holds that $L(\mathcal{F}) - L(\mathcal{F}') \geq -1$ and $\mu(\mathcal{F}) - \mu(\mathcal{F}') \geq \epsilon$. So we have

$$M(\mathcal{F}) - M(\mathcal{F}') = L(\mathcal{F}) - L(\mathcal{F}') + 2(\mu(\mathcal{F}) - \mu(\mathcal{F}'))/\epsilon \geq -1 + 2 = 1.$$

Since each reduction rule decreases $M(\mathcal{F})$ by at least 1, it must stop in polynomial time if we iteratively apply the reduction rules. $\square$

6.2. Step 3

Step 3. If the degree of $\mathcal{F}$ is at least 6, select a variable x with the maximum degree and return $\text{SAT}(\mathcal{F}_{x=1}) \vee \text{SAT}(\mathcal{F}_{x=0})$.

In this step, we branch on a variable x of degree at least 6. The two sub-branches are: $S_1 = \{x\}$; $S_2 = \{\bar{x}\}$. We have the following result:

Lemma 16. *The branching vector generated by Step 3 is covered by*

$$[w_6 + \delta_6, w_6 + 11\delta_6]. \quad (10)$$

Proof. Since R-Rule 4 is not applicable, both x and $\bar{x}$ are $(1^+, 1^+)$ -literals in $\mathcal{F}$. Assume x is a $(j, d - j)$ -literal with $j \geq 1$. By Lemma 8 and $\delta_d = \delta_6$ for $d \geq 6$, we get that

$$\Delta_{S_1} \geq \xi_{S_1}^{(1)} \geq w_d + j\delta_d \geq w_6 + \delta_6$$

Similarly, we have $\Delta_{S_2} \geq w_6 + \delta_6$.

By Lemma 10, $w_3 > \delta_d$ and $w_i > 2\delta_d$ for $4 \leq i \leq d$, we have that

$$\begin{aligned} \Delta_{S_1} + \Delta_{S_2} &\geq \Delta_{S_1}^* + \Delta_{S_2}^* \geq 2w_d + 2d\delta_d + (n'_3(x) + n'_3(\bar{x}))(2w_3 - 2\delta_d) + \sum_{4 \leq i \leq d} (n'_i(x) + n'_i(\bar{x}))(w_i - 2\delta_d) \\ &\geq 2w_6 + 12\delta_d. \end{aligned}$$

Since $\delta_d = \delta_6$ for $d \geq 6$, we obtain

$$\Delta_{S_1} + \Delta_{S_2} \geq 2w_6 + 12\delta_6.$$

As $\min(\Delta_{S_1}, \Delta_{S_2}) \geq w_6 + \delta_6$ and $\Delta_{S_1} + \Delta_{S_2} \geq 2w_6 + 12\delta_6$, by Lemma 6, we know that the branching vector generated by this step is covered by

$$[w_6 + \delta_6, w_6 + 11\delta_6]. \quad \square$$

6.3. Step 4

Step 4. If there is a $(1, 4)$ -literal x (assume that xC is the unique clause containing x), return $\text{SAT}(\mathcal{F}_{x=1} \& C=0) \vee \text{SAT}(\mathcal{F}_{x=0})$.

After Step 3, the degree of $\mathcal{F}$ is at most 5. In this step, the algorithm will branch on a $(1, 4)$ -literal x . The two sub-branches are: $S_1 = \{x\} \cup \bar{C}$; $S_2 = \{\bar{x}\}$. We have the following result:

Lemma 17. *The branching vector generated by Step 4 is covered by*

$$[w_5 + 2w_3, w_5 + w_3 + 7\delta_5]. \quad (11)$$

Proof. By Lemma 11, we get

$$\Delta_{S_1} + \Delta_{S_2} \geq 2w_5 + 3w_3 + 7\delta_5 \quad \text{and} \quad \min(\Delta_{S_1}, \Delta_{S_2}) \geq w_5 + 2w_3.$$

By Lemma 6, we know that the branching vector of this step is covered by

$$[w_5 + 2w_3, w_5 + w_3 + 7\delta_5]. \quad \square$$

6.4. Step 5

Step 5. If there is a 5-literal x such that at least two 2-clauses contain x or $\bar{x}$, return $\text{SAT}(\mathcal{F}_{x=1}) \vee \text{SAT}(\mathcal{F}_{x=0})$.

Note that after Step 4, x is either a $(2, 3)$ -literal or $(3, 2)$ -literal. In this step, the two sub-branches are: $S_1 = \{x\}$; $S_2 = \{\bar{x}\}$. We have the following result:

Lemma 18. *The branching vector generated by Step 5 is covered by*

$$[w_5 + 2\delta_5, w_5 + 4w_3 + 4\delta_5]. \quad (12)$$

Proof. Since x is a $(j, 5 - j)$ -literal with $2 \leq j \leq 3$, by Lemma 8, we have

$$\Delta_{S_1} \geq \xi_{S_1}^{(1)} \geq w_5 + 2\delta_5.$$

Similarly, we can get $\Delta_{S_2} \geq w_5 + 2\delta_5$.

By Lemma 10, $w_5 \geq w_4 \geq 2w_3$, and $\sum_{3 \leq i \leq 5} n'_i(x) + n'_i(\bar{x}) \geq 2$, we have

$$\Delta_{S_1} + \Delta_{S_2} \geq \Delta_{S_1}^* + \Delta_{S_2}^* \geq 2w_5 + 2 \cdot 5\delta_5 + (n'_3(x) + n'_3(\bar{x}))(2w_3 - 2\delta_5) + \sum_{4 \leq i \leq 5} (n'_i(x) + n'_i(\bar{x}))(w_i - 2\delta_5).$$

Note that $w_5 \geq w_4 = 2w_3$ by (2). We can get $w_i - 2\delta_5 \geq 2w_3 - 2\delta_5$ for $3 \leq i \leq 4$. So we have

$$\Delta_{S_1} + \Delta_{S_2} \geq 2w_5 + 10\delta_5 + \sum_{3 \leq i \leq 5} (n'_i(x) + n'_i(\bar{x}))(2w_3 - 2\delta_5).$$

Since there are at least two 2-clauses containing literal x or $\bar{x}$, it holds that $\sum_{3 \leq i \leq 5} (n'_i(x) + n'_i(\bar{x})) \geq 2$. We further obtain

$$\begin{aligned} \Delta_{S_1} + \Delta_{S_2} &\geq 2w_5 + 10\delta_5 + \sum_{3 \leq i \leq 5} (n'_i(x) + n'_i(\bar{x}))(2w_3 - 2\delta_5) \\ &\geq 2w_5 + 10\delta_5 + 2(2w_3 - 2\delta_5) \\ &= 2w_5 + 4w_3 + 6\delta_5. \end{aligned}$$

Since $\min(\Delta_{S_1}, \Delta_{S_2}) \geq w_5 + 2\delta_5$, by Lemma 6, the branching vector of this step is covered by

$$[w_5 + 2\delta_5, w_5 + 4w_3 + 4\delta_5]. \quad \square$$

6.5. Step 6

Step 6. If there are two 5-literals x and y contained in one 2-clause xy , return $\text{SAT}(\mathcal{F}_{x=1}) \vee \text{SAT}(\mathcal{F}_{x=0})$.

After Step 5, a 5-variable can be contained in at most one 2-clause. In this step, if there is a 2-clause xy containing two 5-variables, we pick one of the 5-variables, say x , and branch on it. The two sub-branches are: $S_1 = \{x\}$; $S_2 = \{\bar{x}\}$.

This case will not be the bottleneck case in our algorithm. We will show that after branching some bottleneck cases, we can always get this step. This implies we can use the shift technique here. We will save a shift $\sigma > 0$ from the branching vector of this step that will be included in some bad branching vectors. So in our following analysis, both Δ_{S_1} and Δ_{S_2} should be subtracted by σ . The value of σ will be decided later. We have the following result for this step.

Lemma 19. *The branching vector generated by Step 6 is covered by*

$$[w_5 + 3\delta_5 - \sigma, 2w_5 + 2w_3 + 3\delta_5 - \sigma] \quad (13)$$

Proof. Assume that x is a $(j, 5-j)$ -literal, where $j = 2$ or 3 . Since x is contained in only one 2-clause and there is no 1-clause now, we have that $\sum_{i \geq 3} n_i(x) \geq 2j - 1$. By Lemma 7, we get

$$\Delta_{S_1} \geq \xi_{S_1}^{(1)} \geq w_5 + \sum_{3 \leq i \leq 5} n_i(x)\delta_5 \geq w_5 + (2j - 1)\delta_5.$$

Since we save a shift $\sigma > 0$ from this step, we have

$$\Delta_{S_1} \geq \xi_{S_1}^{(1)} \geq w_5 + (2j - 1)\delta_5 - \sigma.$$

Next, we analyze Δ_{S_2} . By Lemma 7, we first get

$$\xi_{S_2}^{(1)} \geq w_5 + \sum_{3 \leq i \leq 5} n_i(\bar{x})\delta_i.$$

We look at $\mathcal{F}_{S_2=1}$, which is the formula after assigning 1 to $\bar{x}$ in $\mathcal{F}$. By Lemma 3 and Lemma 4, we know that in $\mathcal{F}_{S_2=1}$, $\text{var}(y)$ is a variable of degree at least 4 and there is a 1-clause $\{y\}$. Let $P = \{z : z \in N(y, \mathcal{F}) \text{ and } z \notin \{x, \bar{x}\}\}$ and $Q = \{z : z \in N(y, \mathcal{F}_{S_2=1}) \text{ and } \deg(z) \geq 3 \text{ in } \mathcal{F}_{S_2=1}\}$. By applying R-Rule 4, we will assign 1 to y and remove the literals in $N(y, \mathcal{F}_{S_2=1})$, which may further reduce the measure. Thus, we have $\xi_{S_2}^{(2)} \geq w_4 + |Q|\delta_5$ since $\delta_5 \leq \delta_4 \leq \delta_3$. We get that

$$\Delta_{S_2} \geq \xi_{S_2}^{(1)} + \xi_{S_2}^{(2)} \geq w_5 + \sum_{3 \leq i \leq 5} n_i(\bar{x})\delta_i + w_4 + |Q|\delta_5.$$

Note that $Q \subseteq P$. Let $T = \{l : l \in N(\bar{x}, \mathcal{F}) \text{ and } \deg(l) \leq 4 \text{ in } \mathcal{F}\}$, then it holds that $P \setminus Q \subseteq T$. This is due to that if there is a literal $z \in P \setminus Q$, then z must be a neighbor of $\bar{x}$ in $\mathcal{F}$. By Lemma 4, we know that both z and $\bar{z}$ would appear at most once in $N(\bar{x}, \mathcal{F})$, and so the degree of z is at most 4 in $\mathcal{F}$. Thus, we have

$$|P| - |Q| = |P \setminus Q| \leq |\{l : l \in N(\bar{x}, \mathcal{F}) \text{ and } \deg(l) \leq 4 \text{ in } \mathcal{F}\}| = n_3(\bar{x}) + n_4(\bar{x}).$$

Since $\bar{x}$ is a $(5-j, j)$ -literal not contained any 2-clause or 1-clause, we have that $\sum_{3 \leq i \leq 5} n_i(\bar{x}) \geq 2(5-j) = 10 - 2j$, which implies $n_5(\bar{x}) \geq 10 - 2j - (n_3(\bar{x}) + n_4(\bar{x}))$. With $n_3(\bar{x}) + n_4(\bar{x}) \geq |P| - |Q|$ and $\delta_3 = \delta_4 = w_3$, we further get

$$\begin{aligned}
\Delta_{S_2} &\geq w_5 + \sum_{3 \leq i \leq 5} n_i(\bar{x})\delta_i + w_4 + |Q|\delta_5 \\
&\geq 2w_5 + (n_3(\bar{x}) + n_4(\bar{x}))w_3 + (|Q| - 1 + n_5(\bar{x}))\delta_5 \\
&\geq 2w_5 + (n_3(\bar{x}) + n_4(\bar{x}))w_3 + (|Q| - 1 + 10 - 2j - (n_3(\bar{x}) + n_4(\bar{x})))\delta_5 \\
&= 2w_5 + (n_3(\bar{x}) + n_4(\bar{x}))(w_3 - \delta_5) + (|Q| + 9 - 2j)\delta_5 \\
&\geq 2w_5 + (|P| - |Q|)(w_3 - \delta_5) + (|Q| + 9 - 2j)\delta_5 \\
&= 2w_5 + (9 - 2j)\delta_5 + |P|(w_3 - \delta_5) + |Q|(2\delta_5 - w_3).
\end{aligned}$$

Note that in $\mathcal{F}$, literal y is also a $(2,3)/(3,2)$ -literal contained in exactly one 2-clause xy (since Step 5 has been applied). There is another clause containing y and two different literals z_1 and z_2 , where $\{z_1, z_2\} \cap \{x, \bar{x}\} = \emptyset$ by Lemma 3. So $|P| \geq 2$ holds. With $2\delta_5 > w_3$, we get

$$\Delta_{S_2} \geq 2w_5 + (9 - 2j)\delta_5 + 2(w_3 - \delta_5) \geq 2w_5 + 2w_3 + (7 - 2j)\delta_5.$$

Since a shift $\sigma > 0$ is saved from this step, we have

$$\Delta_{S_2} \geq 2w_5 + 2w_3 + (7 - 2j)\delta_5 - \sigma.$$

By summing Δ_{S_1} and Δ_{S_2} (with shift) up, we have

$$\begin{aligned}
\Delta_{S_1} + \Delta_{S_2} &\geq w_5 + (2j - 1)\delta_5 - \sigma + 2w_5 + 2w_3 + (7 - 2j)\delta_5 - \sigma \\
&= 3w_5 + 2w_3 + 6\delta_5 - 2\sigma.
\end{aligned}$$

Since $2 \leq j \leq 3$, we have $\Delta_{S_1} \geq w_5 + (2j - 1)\delta_5 - \sigma \geq w_5 + 3\delta_5 - \sigma$ and $\Delta_{S_2} \geq 2w_5 + 2w_3 + (7 - 2j)\delta_5 - \sigma \geq 2w_5 + 2w_3 + \delta_5 + \sigma$. Recall that $w_3 \geq \delta_5$, it holds that $\min(\Delta_{S_1}, \Delta_{S_2}) \geq \min(w_5 + 3\delta_5 - \sigma, 2w_5 + 2w_3 + \delta_5 - \sigma) \geq w_5 + 3\delta_5 - \sigma$ since $(2w_5 + 2w_3 + \delta_5 - \sigma) - (w_5 + 3\delta_5 - \sigma) = w_5 + 2w_3 - 2\delta_5 > 0$. Then by Lemma 6, we know that the branching vector of this step is covered by

$$[w_5 + 3\delta_5 - \sigma, 2w_5 + 2w_3 + 3\delta_5 - \sigma]. \quad \square$$

6.6. Step 7

Step 7. If there is a 5-literal x contained in a 2-clause, return $\text{SAT}(\mathcal{F}_{x=1}) \vee \text{SAT}(\mathcal{F}_{x=0})$.

In this step, if there is a 5-literal x contained in a 2-clause xy , then y must be a 4^- -variable. We branch on x . The two sub-branches are: $S_1 = \{x\}$; $S_2 = \{\bar{x}\}$. We have the following result:

Lemma 20. The branching vector generated by Step 7 is covered by

$$[w_5 + w_3 + 2\delta_5, w_5 + w_3 + 6\delta_5]. \quad (14)$$

Proof. Note that there is at most one 2-clause containing x or $\bar{x}$ after Step 5. All clauses containing x are 3^+ -clauses except clause xy and all clauses containing $\bar{x}$ are 3^+ -clauses. So it holds that

$$\sum_{3 \leq i \leq 5} n_i(x) \geq 3 \text{ and } \sum_{3 \leq i \leq 5} n_i(\bar{x}) \geq 4$$

since both x and $\bar{x}$ are $(2,3)/(3,2)$ -literals.

As y is a 4^- -variable, $n_3(x) + n_4(x) \geq 1$ holds. By Lemma 7 and $w_3 = \delta_3 = \delta_4$, we have

$$\begin{aligned}
\Delta_{S_1} &\geq \xi_{S_1}^{(1)} \geq w_5 + \sum_{3 \leq i \leq 5} n_i(x)\delta_i \\
&= w_5 + (n_3(x) + n_4(x))w_3 + n_5(x)\delta_5 \\
&\geq w_5 + w_3 + 2\delta_5.
\end{aligned}$$

For Δ_{S_2} , by Lemma 7 and $\delta_3 = \delta_4 \geq \delta_5$, we get

$$\Delta_{S_2} \geq \xi_{S_2}^{(1)} \geq w_5 + \sum_{3 \leq i \leq 5} n_i(\bar{x})\delta_i \geq w_5 + \sum_{3 \leq i \leq 5} n_i(\bar{x})\delta_5 \geq w_5 + 4\delta_5.$$

By the condition of this step, we have

$$\sum_{3 \leq i \leq 4} n'_i(x) = 1 \text{ and } n'_5(x) + n'_5(\bar{x}) = 0.$$

By Lemma 10 and $w_4 = 2w_3$, we have

$$\begin{aligned} \Delta_{S_1} + \Delta_{S_2} &\geq \Delta_{S_1}^* + \Delta_{S_2}^* \\ &\geq 2w_5 + 2 \cdot 5\delta_5 + (n'_3(x) + n'_3(\bar{x}))(2w_3 - 2\delta_5) + \sum_{4 \leq i \leq 5} (n'_i(x) + n'_i(\bar{x}))(w_i - 2\delta_5) \\ &\geq 2w_5 + 10\delta_5 + (2w_3 - 2\delta_5) \\ &= 2w_5 + 2w_3 + 8\delta_5. \end{aligned}$$

Since $\min(\Delta_{S_1}, \Delta_{S_2}) \geq w_5 + 3\delta_5$ and $\Delta_{S_1} + \Delta_{S_2} \geq 2w_5 + 2w_3 + 8\delta_5$, by Lemma 6, we know that the branching vector of this step is covered by

$$[w_5 + w_3 + 2\delta_5, w_5 + w_3 + 6\delta_5]. \quad \square$$

Lemma 21. After Step 7, if we branch on a 5-literal x and the two sub-branches are $S_1 = \{x\}$ and $S_2 = \{\bar{x}\}$, then it holds that:

$$\sum_{3 \leq i \leq 5} (n_i(x) + n_i(\bar{x})) \geq 10 \text{ and } \min(\Delta_{S_1}, \Delta_{S_2}) \geq w_5 + 4\delta_5.$$

Proof. Note that after Step 7, all clauses containing x or $\bar{x}$ are 3^+ -clauses. So it holds that

$$\sum_{3 \leq i \leq 5} (n_i(x) + n_i(\bar{x})) \geq 5 \cdot 2 = 10.$$

We also have $\sum_{3 \leq i \leq 5} n_i(x) \geq 4$ since x is a $(2, 3)/(3, 2)$ -literal. By Lemma 7 and $\delta_3 = \delta_4 \geq \delta_5$, we have

$$\Delta_{S_1} \geq \xi_S^{(1)} \geq w_5 + \sum_{3 \leq i \leq 5} n_i(x)\delta_i \geq w_5 + \sum_{3 \leq i \leq 5} n_i(x)\delta_5 \geq w_5 + 4\delta_5.$$

We can also get $\Delta_{S_2} \geq w_5 + 4\delta_5$ in a similar way. Thus the lemma holds. $\square$

The above lemma shows some properties after Step 7. We will use it in the next several subsections, and we focus on analyzing the lower bound of $\Delta_{S_1} + \Delta_{S_2}$ to get the branching vectors of Steps 8-12.

6.7. Step 8

Step 8. If there is a 5-literal x such that $N(x, \mathcal{F})$ and $N(\bar{x}, \mathcal{F})$ contain at least two 4^- -literals, return $\text{SAT}(\mathcal{F}_{x=1}) \vee \text{SAT}(\mathcal{F}_{x=0})$.

After Step 7, all clauses containing x or $\bar{x}$ are 3^+ -clauses. In this step, we branch on a variable x such that there are at least two literals of 4^- -variables in $N(x, \mathcal{F})$ and $N(\bar{x}, \mathcal{F})$. The two sub-branches are: $S_1 = \{x\}$; $S_2 = \{\bar{x}\}$. We have the following result:

Lemma 22. The branching vector generated by Step 8 is covered by

$$[w_5 + 4\delta_5, w_5 + 2w_3 + 4\delta_5]. \quad (15)$$

Proof. By Lemma 21 and the condition of this case, we have

$$\sum_{3 \leq i \leq 5} (n_i(x) + n_i(\bar{x})) \geq 10 \text{ and } \sum_{3 \leq i \leq 4} (n_i(x) + n_i(\bar{x})) \geq 2.$$

By Lemma 13, we have

$$\Delta_{S_1} + \Delta_{S_2} \geq \xi_{S_1}^{(1)} + \xi_{S_2}^{(1)} \geq 2w_5 + 10\delta_5 + 2(w_3 - \delta_5) = 2w_5 + 8\delta_5 + 2w_3.$$

As $\min(\Delta_{S_1}, \Delta_{S_2}) \geq w_5 + 4\delta_5$ by Lemma 21, by Lemma 6 we know that the branching vector of this step is covered by

$$[w_5 + 4\delta_5, w_5 + 2w_3 + 4\delta_5]. \quad \square$$

Lemma 23. Let $\mathcal{F}$ be a reduced CNF-formula. After Step 8, if $\mathcal{F}_{x=1} \neq \mathcal{F}'_{x=1}$ for a $(2, 3)/(3, 2)$ -literal $x \in \mathcal{F}$, then it holds that

$$\mu(\mathcal{F}) - \mu(\mathcal{F}'_{x=1}) \geq w_3 - 1.$$

In other words, if we can apply reduction rules on $\mathcal{F}_{x=1}$, the measure would decrease by at least $w_3 - 1$.

Proof. R-Rules 1-8 only remove some literals, and so applying any one of them decreases the measure by at least δ_5 .

After Step 7, all clauses containing variable x are 3^+ -clauses, some 2-clauses would be generated in $\mathcal{F}_{x=1}$, and so R-Rule 9 may be applicable in $\mathcal{F}_{x=1}$ if R-Rules 1-8 do not apply. After Step 8, there is at most one 4^- -variable in $N(x, \mathcal{F})$ and $N(\bar{x}, \mathcal{F})$, and so all 2-clauses in $\mathcal{F}_{x=1}$ contain at least one 5-variable. Thus applying R-Rule 9 decreases the measure by at least $\min\{w_5 + w_i - w_{5+i-2} \mid 3 \leq i \leq 5\} = w_3 - 1$.

For R-Rule 10, we claim that if R-Rules 1-9 are not applicable on $\mathcal{F}_{x=1}$, then R-Rule 10 is also not applicable. The reason is as follows. If there exists two clauses CD_1 and CD_2 in $\mathcal{F}_{x=1}$ such that R-Rule 10 is applicable on $\mathcal{F}_{x=1}$, then there must be two clauses $C'D'_1$ and $C'D'_2$ in $\mathcal{F}$ such that $C \subseteq C'$, $D_1 \subseteq D'_1$, and $D_2 \subseteq D'_2$ since $\mathcal{F}_{x=1}$ is obtained by removing some literals and clauses from $\mathcal{F}$. This implies we could apply R-Rule 10 on $\mathcal{F}$, which contradicts the condition that $\mathcal{F}$ is reduced.

Thus, it holds that either $\mathcal{F}_{x=1} = \mathcal{F}'_{x=1}$ or $\mu(\mathcal{F}) - \mu(\mathcal{F}'_{x=1}) \geq \min\{\delta_5, w_3 - 1\} = w_3 - 1$ by the assumption in (5). $\square$

Recall that after Step 8, any 5-literal x must be a $(2, 3)/(3, 2)$ -literal. We assume $xC_1, xC_2, \bar{x}D_1, \bar{x}D_2$, and $\bar{x}D_3$ are the five clauses containing x or $\bar{x}$ in the analysis of Step 9 and Step 10.

6.8. Step 9

Step 9. If there exist 5-literals y_1 and y_2 such that $y_1 \in C_1$, $y_1 \in D_1$, $y_2 \in C_2$ and y_2 or $\bar{y}_2 \in D_2$, return $\text{SAT}(\mathcal{F}_{y_1=1}) \vee \text{SAT}(\mathcal{F}_{y_1=0})$.

In this step, the two sub-branches are: $S_1 = \{y_1\}$; $S_2 = \{\bar{y}_1\}$. We have the following result:

Lemma 24. The branching vector generated by Step 9 is covered by

$$[w_5 + 4\delta_5, w_5 + \delta_4 + 6\delta_5]. \quad (16)$$

Proof. Note that $x, \bar{x} \in N(y_1, \mathcal{F})$, and so $\text{var}(y_1)$ is a 5-variable. Otherwise we can do Step 8. This implies $t_{5,2}(y_1) \geq 1$.

By Lemma 12, we have

$$\begin{aligned} \xi_{S_1}^{(1)} + \xi_{S_2}^{(1)} &\geq 2w_5 + \sum_{3 \leq i \leq 5} (n_i(y_1) + n_i(\bar{y}_1))\delta_i + (t_{5,2}(y_1) + t_{5,2}(\bar{y}_1))(\delta_4 - \delta_5) \\ &\geq 2w_5 + \sum_{3 \leq i \leq 5} (n_i(y_1) + n_i(\bar{y}_1))\delta_5 + (t_{5,2}(y_1) + t_{5,2}(\bar{y}_1))(\delta_4 - \delta_5) \\ &\geq 2w_5 + 10\delta_5 + (\delta_4 - \delta_5) \\ &= 2w_5 + \delta_4 + 9\delta_5. \end{aligned}$$

Since R-Rule 10 is not applicable, literal y_1 would not appear in C_2 , D_2 and D_3 . We look at $\mathcal{F}_{S_1=1}$, which is the resulting formula after we assign value 1 to y_1 . In $\mathcal{F}_{S_1=1}$, x becomes a 3-variable, the three clauses containing x or $\bar{x}$ will be $xC_2, \bar{x}D_2$, and $\bar{x}D_3$, and $y_2 \in C_2$.

Case 1. If $y_2 \in D_2$, then R-Rule 7 is applicable. Applying any one of R-Rules 1-7 decreases the measure by at least δ_5 since each of them removes at least one literal.

Case 2. If $\bar{y}_2 \in D_2$, then R-Rule 5 is applicable. Resolution on x decreases the measure by w_3 .

So it holds that $\xi_{S_1}^{(2)} + \xi_{S_1}^{(3)} \geq \min(\delta_5, w_3) = \delta_5$ and we have

$$\Delta_{S_1} + \Delta_{S_2} \geq \xi_{S_1}^{(1)} + \xi_{S_2}^{(1)} + (\xi_{S_1}^{(2)} + \xi_{S_1}^{(3)}) \geq 2w_5 + \delta_4 + 10\delta_5.$$

By Lemma 21, we have $\min(\Delta_{S_1}, \Delta_{S_2}) \geq w_5 + 4\delta_5$. By Lemma 6, we know that the branching vector of this step is covered by

$$[w_5 + 4\delta_5, w_5 + \delta_4 + 6\delta_5]. \quad \square$$

6.9. Step 10

Step 10. If there exist 5-literals y_1 and y_2 such that $y_1 \in C_1$, $\bar{y}_1 \in D_1$, $y_2 \in C_2$, and $\bar{y}_2 \in D_2$, pick a 5-literal $z \in D_3$ and return $\text{SAT}(R_5(\mathcal{F}_{z=1})) \vee \text{SAT}(\mathcal{F}_{z=0})$, where $R_5(\mathcal{F}_{z=1})$ is the resulting formula after applying R-Rule 5 once on $\mathcal{F}_{z=1}$.

After Step 7, all clauses containing 5-literals are 3^+ -clauses so $|D_3| \geq 2$. After Step 8, there is at most one 4^- -literal in D_3 . So there must exist a 5-literal $z \in D_3$, we branch on this literal and the two sub-branches are: $S_1 = \{z\}$; $S_2 = \{\bar{z}\}$. Note that we will first apply R-Rule 5 on $\mathcal{F}_{S_1=1}$. We have the following result:

Lemma 25. *The branching vector generated by Step 10 is covered by*

$$[w_5 + 4\delta_5, w_5 + w_4 + 6\delta_5]. \quad (17)$$

Proof. By Lemma 13 with $g \geq 10$ (since x is not contained in any 2-clause after Step 7), we get

$$\xi_{S_1}^{(1)} + \xi_{S_2}^{(1)} \geq 2w_5 + 10\delta_5.$$

Note that literal z would not appear in C_1 and C_2 , otherwise Step 9 would be applied. We look at $\mathcal{F}_{S_1=1}$, which is the resulting formula after we assign value 1 to z . In $\mathcal{F}_{S_1=1}$, x becomes a 4-variable and the four clauses containing x or $\bar{x}$ are xC_1 , xC_2 , $\bar{x}D_1$, and $\bar{x}D_2$. Since $y_1 \in C_1$, $y_2 \in C_2$, $\bar{y}_1 \in D_1$, and $\bar{y}_2 \in D_2$, we can apply R-Rule 5 (resolution on x). This decreases the measure by w_4 . Thus $\xi_{S_1}^{(3)} \geq w_4$ and we have

$$\Delta_{S_1} + \Delta_{S_2} \geq \xi_{S_1}^{(1)} + \xi_{S_2}^{(1)} + \xi_{S_1}^{(3)} \geq 2w_5 + 10\delta_5 + w_4.$$

With $\min(\Delta_{S_1}, \Delta_{S_2}) \geq w_5 + 4\delta_5$ by Lemma 21, by Lemma 6 we have that the branching vector of this step is covered by

$$[w_5 + 4\delta_5, w_5 + w_4 + 6\delta_5]. \quad \square$$

For the sake of presentation, we define an auxiliary graph G_x for each literal x as follows.

Definition 3 (Clause-clause incidence graph). Let x be a literal in a formula $\mathcal{F}$. Assume the clauses containing x are $xC_1, xC_2, \dots, xC_a$ and the clauses containing $\bar{x}$ are $\bar{x}D_1, \bar{x}D_2, \dots, \bar{x}D_b$. A *clause-clause incidence graph* of literal x , denoted by G_x , is a bipartite graph with bipartition (X, Y) where X is the set of clauses $C_i (1 \leq i \leq a)$ and Y is the set of clauses $D_j (1 \leq j \leq b)$, and there is an edge between $C_i \in X (1 \leq i \leq a)$ and $D_j \in Y (1 \leq j \leq b)$ if and only if C_i and D_j contain the literal of the same variable in $\mathcal{F}$.

Example. Let $\mathcal{F}$ be a formula and xC and $\bar{x}D$ be two clauses in $\mathcal{F}$. If $y \in C$ and y or $\bar{y} \in D$, then in G_x , there is an edge between vertex C and vertex D .

Lemma 26. *After Step 10, for any $(2, 3)$ -literal x , there is no matching of size at least 2 in G_x .*

Proof. If there exists a matching of size 2 in G_x , then there exists two literals $y_1 \in C_1$ and $y_2 \in C_2$ such that two of $y_1, \bar{y}_1, y_2$, and $\bar{y}_2$ appear in two of D_1, D_2 , and D_3 separately. Thus we would be able to apply Step 9 or Step 10. $\square$

Lemma 27. *After Step 10, for any $(2, 3)$ -literal x , in G_x if all vertices have a degree of at most 2, then there are at least two vertices of degree 0 in G_x .*

Proof. If there is at most one vertex of degree 0, then after deleting the degree-0 vertex, we get a graph with four vertices such that each vertex has a degree of at least 1 and at most 2. For any case, this graph has a matching of size 2, which contradicts Lemma 26. $\square$

6.10. Step 11

Step 11. If there is a 5-literal x contained in at least one 4^+ -clause, return $\text{SAT}(\mathcal{F}_{x=1}) \vee \text{SAT}(\mathcal{F}_{x=0})$.

In this step, we branch on a 5-literal x contained in at least one 4^+ -clause. The two sub-branches are: $S_1 = \{x\}$; $S_2 = \{\bar{x}\}$. The branching vector of this step leads to one of the worst branching factors. But we will prove that the shift $\sigma > 0$ saved in Step 6 (Section 6.5) can be used in this step to get an improvement. We have the following result:

Lemma 28. *The branching vector generated by Step 11 is covered by*

$$[w_5 + 4\delta_5, w_5 + w_3 + 6\delta_5] \text{ or } [w_5 + 4\delta_5, w_5 + 7\delta_5 + \sigma]. \quad (18)$$

Proof. Let $m_4 \geq 1$ be the number of 4^+ -clauses containing literal x or $\bar{x}$. Then the number of 3-clauses containing x or $\bar{x}$ is $5 - m_4$. We have

$$\sum_{3 \leq i \leq 5} (n_i(x) + n_i(\bar{x})) \geq 2(5 - m_4) + 3m_4 \geq 10 + m_4.$$

Next, we consider several cases. By Lemma 21, we have $\min(\Delta_{S_1}, \Delta_{S_2}) \geq w_5 + 4\delta_5$ for all the following cases. So we focus on analyzing $\Delta_{S_1} + \Delta_{S_2}$.

Case 1. There is a variable y such that both y and $\bar{y}$ appear in $N(x, \mathcal{F})$ or $N(\bar{x}, \mathcal{F})$.

Note that after Step 8, there is at most one 4^- -literal in $N(x, \mathcal{F})$ or $N(\bar{x}, \mathcal{F})$, and so we have $t_{5,2}(x) + t_{5,2}(\bar{x}) \geq 1$. By Lemma 13 with $g = 10 + m_4$, we have

$$\xi_{S_1}^{(1)} + \xi_{S_2}^{(1)} \geq 2w_5 + (10 + m_4)\delta_5 + 1 \cdot (\delta_4 - \delta_5) \geq 2w_5 + w_3 + 10\delta_5.$$

By Lemma 6, we know that the branching vector of this case is covered by

$$[w_5 + 4\delta_5, w_5 + w_3 + 6\delta_5].$$

Case 2. There are 4^- -literals in $N(x, \mathcal{F})$ or $N(\bar{x}, \mathcal{F})$, i.e., $\sum_{3 \leq i \leq 4} (n_i(x) + n_i(\bar{x})) \geq 1$. By Lemma 13 with $g = 10 + m_4$ and $h = 1$, we have

$$\begin{aligned} \Delta_{S_1} + \Delta_{S_2} &\geq \xi_{S_1}^{(1)} + \xi_{S_2}^{(1)} \geq 2w_5 + (10 + m_4)\delta_5 + 1 \cdot (w_3 - \delta_5) \\ &\geq 2w_5 + w_3 + 10\delta_5. \end{aligned}$$

By Lemma 6, the branching vector of this case is covered by

$$[w_5 + 4\delta_5, w_5 + w_3 + 6\delta_5].$$

Case 3. There are no 4^- -literals in $N(x, \mathcal{F})$ or $N(\bar{x}, \mathcal{F})$ and no variables y such that both y and $\bar{y}$ appear in $N(x, \mathcal{F})$ or $N(\bar{x}, \mathcal{F})$, i.e., $\sum_{3 \leq i \leq 4} (n_i(x) + n_i(\bar{x})) = 0$ and $t_{5,2}(x) + t_{5,2}(\bar{x}) = 0$.

Case 3.1. There are at least two 4^+ -clauses containing x or $\bar{x}$, i.e., $m_4 \geq 2$. By Lemma 13 with $g = 10 + m_4$, we have

$$\Delta_{S_1} + \Delta_{S_2} \geq \xi_{S_1}^{(1)} + \xi_{S_2}^{(1)} \geq 2w_5 + (10 + m_4)\delta_5 \geq 2w_5 + 12\delta_5.$$

Since $w_3 < 2\delta_5$, the branching vector of this case is covered by that of Case 1.

Case 3.2. There is only one 4^+ -clause containing variable x , i.e., $m_4 = 1$. Similar to case 2.1, by Lemma 13 with $g = 10 + m_4$, we have

$$\xi_{S_1}^{(1)} + \xi_{S_2}^{(1)} \geq 2w_5 + (10 + m_4)\delta_5 = 2w_5 + 11\delta_5.$$

Next, we consider $\mathcal{F}'_{x=1}$ and $\mathcal{F}'_{x=0}$.

Case 3.2.1. Some reduction rules can be applied on $\mathcal{F}_{x=1}$ or $\mathcal{F}_{x=0}$.

By Lemma 23, the total measure will further decreases by at least $w_3 - 1$ and we have $\Delta_{S_1} + \Delta_{S_2} \geq 2w_5 + 11\delta_5 + w_3 - 1$. Since $\delta_5 \geq 1$, the branching vector of this case is covered by that of Case 1.

Case 3.2.2. $\mathcal{F}'_{x=1} = \mathcal{F}_{x=1}$ and $\mathcal{F}'_{x=0} = \mathcal{F}_{x=0}$.

We show that we can apply Step 6 (Section 6.5) on either $\mathcal{F}'_{x=1}$ or $\mathcal{F}'_{x=0}$ to use the saved shift σ . Look at G_x and consider the following two cases.

(1) There is a vertex of degree at least 3 in G_x . Recall that the five clauses containing literal x are $xC_1, xC_2, \bar{x}D_1, \bar{x}D_2$, and $\bar{x}D_3$. If there is a vertex in G_x with degree at least 3, the corresponding clause of this vertex must be C_1 or C_2 , w.l.o.g., let us assume that the clause is C_1 . By the condition of Case 3 and $\mathcal{F}$ being reduced, for a literal $y \in C_1$, at most one of y and $\bar{y}$ will appear in D_1, D_2 , and D_3 . So if C_1 is a degree-3 vertex in G_x , it must contain at least three different literals. By the condition of Case 3.2, we know xC_1 is the unique 4^+ -clause containing variable x , and so $|C_2| = 2$. By Lemma 26, in G_x the degree of vertex C_2 is 0. There are no 4^- -literals in C_2 by the condition of Case 3. Thus, in $\mathcal{F}'_{x=0}$, clause C_2 contains two 5-variables, and so we can apply Step 6 on $\mathcal{F}'_{x=0}$.

(2) All vertices in G_x have a degree of at most 2. By Lemma 27 there are at least two vertices in G_x with degree 0. By the condition of Case 3, there are no 4^- -literals in those clauses, and then we will get some 2-clause containing two 5-literals in $\mathcal{F}'_{x=1}$ or $\mathcal{F}'_{x=0}$ and we can further apply Step 6 on $\mathcal{F}'_{x=1}$ or $\mathcal{F}'_{x=0}$.

Thus, we further get

$$\Delta_{S_1} + \Delta_{S_2} \geq \xi_{S_1}^{(1)} + \xi_{S_2}^{(1)} + \sigma = 2w_5 + 11\delta_5 + \sigma.$$

Note that $\min(\Delta_{S_1}, \Delta_{S_2}) \geq w_5 + 4\delta_5$. By Lemma 6, the branching vector of this case is covered by

$$[w_5 + 4\delta_5, w_5 + 7\delta_5 + \sigma].$$

In summary, the branching vector is covered by

$$[w_5 + 4\delta_5, w_5 + w_3 + 6\delta_5] \text{ or } [w_5 + 4\delta_5, w_5 + 7\delta_5 + \sigma]. \quad \square$$

6.11. Step 12

Step 12. If there is a clause containing both a 5-literal x and a 4⁻-literal, return $\text{SAT}(\mathcal{F}_{x=1}) \vee \text{SAT}(\mathcal{F}_{x=0})$.

After Step 11, all clauses containing x or $\bar{x}$ are 3-clauses. In this step, we branch on a 5-literal x such that there is one 4⁻-literal in $N(x, \mathcal{F})$. The two sub-branches are: $S_1 = \{x\}$; $S_2 = \{\bar{x}\}$. Similar to Step 11, the shift $\sigma > 0$ saved in Step 6 (Section 6.5) will be used in this step. We have the following result:

Lemma 29. *The branching vector generated by Step 12 is covered by*

$$[w_5 + 4\delta_5, w_5 + 4\delta_5 + 2w_3] \text{ or } [w_5 + 4\delta_5, w_5 + w_3 + 5\delta_5 + \sigma]. \quad (19)$$

Proof. By the condition of this step, we have $\sum_{3 \leq i \leq 4} (n_i(x) + n_i(\bar{x})) \geq 1$. Next, we consider two cases. For all of the following cases, we have $\min(\Delta_{S_1}, \Delta_{S_2}) \geq w_5 + 4\delta_5$ by Lemma 21. So we focus on $\Delta_{S_1} + \Delta_{S_2}$.

Case 1. There is a variable y such that both y and $\bar{y}$ appear in $N(x, \mathcal{F})$ or $N(\bar{x}, \mathcal{F})$.

Note that after Step 8, there is at most one 4⁻-literal in $N(x, \mathcal{F})$ or $N(\bar{x}, \mathcal{F})$, and so we have $t_{5,2}(x) + t_{5,2}(\bar{x}) \geq 1$. By Lemma 13 and $\delta_4 = w_3$, we have

$$\xi_{S_1}^{(1)} + \xi_{S_2}^{(1)} \geq 2w_5 + 10\delta_5 + 1 \cdot (w_3 - \delta_5) + 1 \cdot (\delta_4 - \delta_5) = 2w_5 + 8\delta_5 + 2w_3.$$

By Lemma 6, the branching vector of this case is covered by

$$[w_5 + 4\delta_5, w_5 + 4\delta_5 + 2w_3].$$

Case 2. There are no variables y such that both y and $\bar{y}$ appear in $N(x, \mathcal{F})$ or $N(\bar{x}, \mathcal{F})$.

By Lemma 13, we first have

$$\xi_{S_1}^{(1)} + \xi_{S_2}^{(1)} \geq 2w_5 + 10\delta_5 + 1 \cdot (w_3 - \delta_5) = 2w_5 + w_3 + 9\delta_5.$$

Similar to Step 11, we consider $\mathcal{F}'_{x=1}$ and $\mathcal{F}'_{x=0}$.

Case 2.1. Some reduction rules can be applied on $\mathcal{F}_{x=1}$ or $\mathcal{F}_{x=0}$.

By Lemma 23 the total measure will further decrease by at least $w_3 - 1$ and we have $\Delta_{S_1} + \Delta_{S_2} \geq 2w_5 + 9\delta_5 + 2w_3 - 1$.

By Lemma 6, the branching vector of this case is covered by

$$[w_5 + 4\delta_5, w_5 + 5\delta_5 + 2w_3 - 1].$$

Since $\delta_5 \geq 1$, this branching vector is covered by that of Case 1.

Case 2.2. $\mathcal{F}'_{x=1} = \mathcal{F}_{x=1}$ and $\mathcal{F}'_{x=0} = \mathcal{F}_{x=0}$.

We look at the auxiliary graph G_x . In G_x , all the vertices have a degree of at most 2 since all clauses containing x or $\bar{x}$ are 3-clauses and no variables y such that both y and $\bar{y}$ appear in $N(x, \mathcal{F})$ or $N(\bar{x}, \mathcal{F})$. With Lemma 27, there are at least two vertices with degree 0 in G_x . Let the set of corresponding clauses of them be E . Since there may be at most one clause in E that contains a 4⁻-literal, there must exist a 2-clause containing two 5-literals in $\mathcal{F}'_{x=1}$ or $\mathcal{F}'_{x=0}$, and thus we can apply Step 6 on $\mathcal{F}'_{x=1}$ or $\mathcal{F}'_{x=0}$. So we further get

$$\Delta_{S_1} + \Delta_{S_2} \geq 2w_5 + w_3 + 9\delta_5 + \sigma.$$

By Lemma 6, the branching vector of this case is covered by

$$[w_5 + 4\delta_5, w_5 + w_3 + 5\delta_5 + \sigma].$$

In summary, the branching vector of this step is covered by

$$[w_5 + 4\delta_5, w_5 + 4\delta_5 + 2w_3] \text{ or } [w_5 + 4\delta_5, w_5 + w_3 + 5\delta_5 + \sigma]. \quad \square$$

6.12. Step 13

Step 13. If there are still some 5-literals, then $\mathcal{F} = \mathcal{F}_5 \wedge \mathcal{F}_{\leq 4}$, where $\mathcal{F}_5$ is a 3-CNF containing only 5-literals and $\mathcal{F}_{\leq 4}$ contains only 3/4-literals.

In this step, the literals of all 5-variables form a 3-SAT instance $\mathcal{F}_5$. We apply the $O^*(1.32793^n)$ -time algorithm in [6] for 3-SAT to solve our problem, where n is the number of variables in the instance. Since $w_5 = 5$, we have that $n = \mu(\mathcal{F}_5)/w_5 = \mu(\mathcal{F}_5)/5$. So the running time for this part will be

$$O^*(1.32793^{\mu(\mathcal{F}_5)/w_5}) = O^*(1.0584^{\mu(\mathcal{F}_5)}).$$

6.13. Step 14

Step 14. If there is a $(1, 3)$ -literal x (assume that x_C is the unique clause containing x), return $\text{SAT}(\mathcal{F}_{x=1} \& C_{=0}) \vee \text{SAT}(\mathcal{F}_{x=0})$.

After Step 13, all literals in $\mathcal{F}$ are 4^- -literals. In this step, we branch on a $(1, 3)$ -literal x . The two sub-branches are: $S_1 = \{x\}$; $S_2 = \{\bar{x}\}$. We have the following result:

Lemma 30. The branching vector generated by Step 14 is covered by

$$[w_4 + 2w_3, w_4 + 6\delta_4]. \quad (20)$$

Proof. By Lemma 11, we have

$$\Delta_{S_1} + \Delta_{S_2} \geq 2w_4 + 3w_3 + 5\delta_4 = 2w_4 + 8w_3 \quad \text{and} \quad \min(\Delta_{S_1}, \Delta_{S_2}) \geq w_4 + 2w_3.$$

By Lemma 6, the branching vector of this step is covered by

$$[w_4 + 2w_3, w_4 + 6\delta_4]. \quad \square$$

6.14. Step 15

Step 15. If there is a $(2, 2)$ -literal x , return $\text{SAT}(\mathcal{F}_{x=1}) \vee \text{SAT}(\mathcal{F}_{x=0})$.

In this step, the two sub-branches are: $S_1 = \{x\}$; $S_2 = \{\bar{x}\}$. We have the following result:

Lemma 31. The branching vector generated by Step 15 is covered by

$$[w_4 + 2\delta_4, w_4 + 6\delta_4]. \quad (21)$$

Proof. Since both x and $\bar{x}$ are $(2, 2)$ -literals, by Lemma 8 we have

$$\Delta_{S_1} \geq \xi_{S_1}^{(1)} = w_4 + 2\delta_4 \quad \text{and} \quad \Delta_{S_2} \geq \xi_{S_2}^{(1)} \geq w_4 + 2\delta_4.$$

By Lemma 10, we have

$$\begin{aligned} \Delta_{S_1} + \Delta_{S_2} &\geq \Delta_{S_1}^* + \Delta_{S_2}^* \geq 2w_4 + 2 \cdot 4\delta_4 + (n'_3(x) + n'_3(\bar{x}))(2w_3 - 2\delta_4) + (n'_4(x) + n'_4(\bar{x}))(w_4 - 2\delta_4) \\ &= 2w_4 + 8\delta_4. \end{aligned}$$

By Lemma 6, we know that the branching vector is covered by

$$[w_4 + 2\delta_4, w_4 + 6\delta_4]. \quad \square$$

6.15. Step 16

Step 16. Apply the algorithm by Wahlström [32] to solve the instance.

All variables are 3-variables in this step. We apply the $O^*(1.1279^n)$ -time algorithm by Wahlström [32] to solve this special case, where n is the number of variables. For this case, we have $n = \mu(\mathcal{F})/w_3$. So the running time of this part is

$$O^*((1.1279^{1/w_3})^{\mu(\mathcal{F})}).$$

7. The final result

Each of the above branching vectors above will generate a constraint in our quasiconvex program to solve the best value for w_3 , w_4 , and σ . Let α_i denote the branching factor for branching vector (i) where $10 \leq i \leq 21$. We want to find the minimum value α such that $\alpha \geq \alpha_i$ and $\alpha \geq 1.1279^{1/w_3}$ (generated by Step 16) under the assumptions (2) and (5). By solving this quasiconvex program, we get that $\alpha = 1.0638$ by letting $w_3 = 1.94719$, $w_4 = 2w_3 = 3.89438$, and $\sigma = 0.86108$. Note that $\alpha = 1.0638$ is greater than 1.0584, which is the branching factor generated in Step 13. So 1.0638 is the worst branching factor in the whole algorithm. By (6), we get the following result.

Theorem 1. Algorithm 1 solves the SAT problem in $O^*(1.0638^L)$ time.

Table 2
The weight setting.

$w_1 = w_2 = 0$	$\sigma = 0.86108$
$w_3 = 1.94719$	$\delta_3 = 1.94719$
$w_4 = 3.89438$	$\delta_4 = 1.94719$
$w_5 = 5$	$\delta_5 = 1.10562$
$w_i = i (i \geq 6)$	$\delta_i = 1 (i \geq 6)$

Table 3
The branching vector and factor for each step.

Steps	Branching vectors	Factors
Step 3 (Section 6.2)	$[w_6 + \delta_6, w_6 + 11\delta_6]$	1.0636
Step 4 (Section 6.3)	$[w_5 + 2w_3, w_5 + w_3 + 7\delta_5]$	1.0620
Step 5 (Section 6.4)	$[w_5 + 2\delta_5, w_5 + 4w_3 + 4\delta_5]$	1.0624
Step 6 (Section 6.5)	$[w_5 + 3\delta_5 - \sigma, 2w_5 + 2w_3 + 3\delta_5 - \sigma]$	1.0633
Step 7 (Section 6.6)	$[w_5 + w_3 + 2\delta_5, w_5 + w_3 + 6\delta_5]$	1.0638 *
Step 8 (Section 6.7)	$[w_5 + 4\delta_5, w_5 + 2w_3 + 4\delta_5]$	1.0636
Step 9 (Section 6.8)	$[w_5 + 4\delta_5, w_5 + \delta_4 + 6\delta_5]$	1.0629
Step 10 (Section 6.9)	$[w_5 + 4\delta_5, w_5 + w_4 + 6\delta_5]$	1.0584
Step 11 (Section 6.10)	$[w_5 + 4\delta_5, w_5 + w_3 + 6\delta_5]$	1.0629
	$[w_5 + 4\delta_5, w_5 + 7\delta_5 + \sigma]$	1.0629
Step 12 (Section 6.11)	$[w_5 + 4\delta_5, w_5 + 4\delta_5 + 2w_3]$	1.0636
	$[w_5 + 4\delta_5, w_5 + w_3 + 5\delta_5 + \sigma]$	1.0635
Step 13 (Section 6.12)	$O^*((1.32793^{1/w_5})^\mu)$	1.0584
Step 14 (Section 6.13)	$[w_4 + 2w_3, w_4 + 6\delta_4]$	1.0638 *
Step 15 (Section 6.14)	$[w_4 + 2\delta_4, w_4 + 6\delta_4]$	1.0638 *
Step 16 (Section 6.15)	$O^*((1.12791^{1/w_3})^\mu)$	1.0638 *

We also show the whole weight setting in Table 2 and the branching vector of each step under the setting in Table 3.

From Table 3, we can see that we have four bottlenecks (marked by *): Steps 7, 14, 15, and 16. In fact, Steps 14, 15, and 16 have the same branching vector $[4w_3, 8w_3]$ under the assumption that $w_4 = 2w_3$ (for Step 14, the worst branching vector in [32] is $[4, 8]$). The branching factor for these three steps will decrease if the value of w_3 increases. On the other hand, the branching factor for Step 7 will decrease if the value of w_3 decreases. We set the best value of w_3 to balance them. If we can either improve Step 7 or improve Steps 14, 15, and 16 together, then we may get a further improvement.

8. Concluding remarks

In this paper, we show that the SAT problem can be solved in $O^*(1.0638^L)$ time, improving the previous bound in terms of the input length obtained more than ten years ago. Nowadays, improvement becomes harder and harder. However, SAT is one of the most important problems in exact and parameterized algorithms, and the state-of-the-art algorithms are frequently mentioned in the literature. For the techniques, although our algorithm, as well as most previous algorithms, is based on case analyses, we introduce a general analysis framework to get a neat and clear analysis. This framework can even be used to simplify the analysis for other similar algorithms based on the measure-and-conquer method.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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References

- [1] S.A. Cook, D.G. Mitchell, Finding hard instances of the satisfiability problem: a survey, in: D. Du, J. Gu, P.M. Pardalos (Eds.), *Satisfiability Problem: Theory and Applications*, Proceedings of a DIMACS Workshop, Piscataway, New Jersey, USA, March 11–13, 1996, in: DIMACS Series in Discrete Mathematics and Theoretical Computer Science, vol. 35, DIMACS/AMS, 1996, pp. 1–17.
- [2] S.A. Cook, The complexity of theorem-proving procedures, in: M.A. Harrison, R.B. Banerji, J.D. Ullman (Eds.), *Proceedings of the 3rd Annual ACM Symposium on Theory of Computing*, May 3–5, 1971, Shaker Heights, Ohio, USA, ACM, 1971, pp. 151–158.
- [3] A. Biere, M. Heule, H. van Maaren, T. Walsh (Eds.), *Handbook of Satisfiability*, Frontiers in Artificial Intelligence and Applications, vol. 185, IOS Press, 2009.
- [4] R. Impagliazzo, R. Paturi, On the complexity of k-SAT, *J. Comput. Syst. Sci.* 62 (2) (2001) 367–375, <https://doi.org/10.1006/jcss.2000.1727>.
- [5] U. Schöningh, A probabilistic algorithm for k-SAT and constraint satisfaction problems, in: 40th Annual Symposium on Foundations of Computer Science, FOCS '99, 17–18 October, 1999, New York, NY, USA, IEEE Computer Society, 1999, pp. 410–414.
- [6] S. Liu, Chain, generalization of covering code, and deterministic algorithm for k-SAT, in: I. Chatzigiannakis, C. Kaklamanis, D. Marx, D. Sannella (Eds.), 45th International Colloquium on Automata, Languages, and Programming, ICALP 2018, July 9–13, 2018, Prague, Czech Republic, in: LIPIcs, vol. 107, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018, 88.
- [7] R. Paturi, P. Pudlák, F. Zane, Satisfiability coding lemma, *Chic. J. Theor. Comput. Sci.* 1999 (1999), <http://cjcs.cs.uchicago.edu/articles/1999/11/contents.html>.
- [8] R. Paturi, P. Pudlák, M.E. Saks, F. Zane, An improved exponential-time algorithm for k-SAT, *J. ACM* 52 (3) (2005) 337–364, <https://doi.org/10.1145/1066100.1066101>.
- [9] D. Scheder, PPSZ is better than you think, in: 62nd IEEE Annual Symposium on Foundations of Computer Science, FOCS 2021, Denver, CO, USA, February 7–10, 2022, IEEE, 2021, pp. 205–216.
- [10] B. Monien, E. Speckenmeyer, O. Vornberger, Upper bounds for covering problems, *Methods Oper. Res.* 43 (1981) 419–431.
- [11] E.A. Hirsch, Two new upper bounds for SAT, in: H.J. Karloff (Ed.), *Proceedings of the Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*, 25–27 January 1998, San Francisco, California, USA, ACM/SIAM, 1998, pp. 521–530, <http://dl.acm.org/citation.cfm?id=314613.314838>.
- [12] M. Yamamoto, An improved $O(1.234^n)$ -time deterministic algorithm for SAT, in: X. Deng, D. Du (Eds.), *Algorithms and Computation*, 16th International Symposium, Proceedings, ISAAC 2005, Sanya, Hainan, China, December 19–21, 2005, in: Lecture Notes in Computer Science, vol. 3827, Springer, 2005, pp. 644–653.
- [13] H. Chu, M. Xiao, Z. Zhang, An improved upper bound for SAT, *Theor. Comput. Sci.* 887 (2021) 51–62, <https://doi.org/10.1016/j.tcs.2021.06.045>.
- [14] A. Van Gelder, A satisfiability tester for non-clausal propositional calculus, *Inf. Comput.* 79 (1) (1988) 1–21, [https://doi.org/10.1016/0890-5401\(88\)90014-4](https://doi.org/10.1016/0890-5401(88)90014-4).
- [15] O. Kullmann, H. Luckhardt, Deciding propositional tautologies: Algorithms and their complexity, preprint 82 (1997).
- [16] E.A. Hirsch, New worst-case upper bounds for SAT, *J. Autom. Reason.* 24 (4) (2000) 397–420, <https://doi.org/10.1023/A:1006340920104>.
- [17] M. Wahlström, An algorithm for the SAT problem for formulae of linear length, in: G.S. Brodal, S. Leonardi (Eds.), *Algorithms - ESA 2005*, 13th Annual European Symposium, Proceedings, Palma de Mallorca, Spain, October 3–6, 2005, in: Lecture Notes in Computer Science, vol. 3669, Springer, 2005, pp. 107–118.
- [18] J. Chen, Y. Liu, An improved SAT algorithm in terms of formula length, in: F.K.H.A. Dehne, M.L. Gavrilova, J. Sack, C.D. Tóth (Eds.), *Algorithms and Data Structures*, 11th International Symposium, Proceedings, WADS 2009, Banff, Canada, August 21–23, 2009, in: Lecture Notes in Computer Science, vol. 5664, Springer, 2009, pp. 144–155.
- [19] J. Chen, C. Xu, J. Wang, Dealing with 4-variables by resolution: an improved MaxSAT algorithm, *Theor. Comput. Sci.* 670 (2017) 33–44, <https://doi.org/10.1016/j.tcs.2017.01.020>.
- [20] M. Xiao, An exact MaxSAT algorithm: further observations and further improvements, in: L.D. Raedt (Ed.), *Proceedings of the Thirty-First International Joint Conference on Artificial Intelligence, IJCAI 2022*, Vienna, Austria, 23–29 July 2022, ijcai.org, 2022, pp. 1887–1893.
- [21] K. Brilliantov, V. Alferov, I. Bliznets, Improved algorithms for maximum satisfiability and its special cases, in: *Thirty-Seventh AAAI Conference on Artificial Intelligence, AAAI 2023*, AAAI Press, 2023, pp. 3898–3905.
- [22] J. Peng, M. Xiao, A fast algorithm for SAT in terms of formula length, in: C. Li, F. Manyà (Eds.), *Theory and Applications of Satisfiability Testing - SAT 2021 - 24th International Conference*, Proceedings, Barcelona, Spain, July 5–9, 2021, in: Lecture Notes in Computer Science, vol. 12831, Springer, 2021, pp. 436–452.
- [23] F.V. Fomin, D. Kratsch, *Exact Exponential Algorithms*, Texts in Theoretical Computer Science. An EATCS Series, Springer, 2010.
- [24] Y. Iwata, A faster algorithm for dominating set analyzed by the potential method, in: D. Marx, P. Rossmanith (Eds.), *Parameterized and Exact Computation - 6th International Symposium, IPEC 2011*, Saarbrücken, Germany, September 6–8, 2011, in: Lecture Notes in Computer Science, vol. 7112, Springer, 2011, pp. 41–54, Revised Selected Papers.
- [25] J. Chen, I.A. Kanj, G. Xia, Labeled search trees and amortized analysis: improved upper bounds for np-hard problems, *Algorithmica* 43 (4) (2005) 245–273, <https://doi.org/10.1007/s00453-004-1145-7>.
- [26] S. Gaspers, *Exponential Time Algorithms - Structures, Measures, and Bounds*, VDM, 2010.
- [27] M. Xiao, H. Nagamochi, Exact algorithms for maximum independent set, *Inf. Comput.* 255 (2017) 126–146, <https://doi.org/10.1016/j.ic.2017.06.001>.
- [28] F.V. Fomin, F. Grandoni, D. Kratsch, A measure & conquer approach for the analysis of exact algorithms, *J. ACM* 56 (5) (2009) 25, <https://doi.org/10.1145/1552285.1552286>.
- [29] M. Xiao, H. Nagamochi, An exact algorithm for TSP in degree-3 graphs via circuit procedure and amortization on connectivity structure, *Algorithmica* 74 (2) (2016) 713–741, <https://doi.org/10.1007/s00453-015-9970-4>.
- [30] J.M.M. van Rooij, H.L. Bodlaender, Exact algorithms for dominating set, *Discrete Appl. Math.* 159 (17) (2011) 2147–2164, <https://doi.org/10.1016/j.dam.2011.07.001>.
- [31] M. Davis, H. Putnam, A computing procedure for quantification theory, *J. ACM* 7 (3) (1960) 201–215, <https://doi.org/10.1145/321033.321034>.
- [32] M. Wahlström, Faster exact solving of SAT formulae with a low number of occurrences per variable, in: F. Bacchus, T. Walsh (Eds.), *Theory and Applications of Satisfiability Testing*, 8th International Conference, Proceedings, SAT 2005, St. Andrews, UK, June 19–23, 2005, in: Lecture Notes in Computer Science, vol. 3569, Springer, 2005, pp. 309–323.